

# STRENGTHENING TRACK THEORIES

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## 1. INTRODUCTION

Recent work of the first author on the homotopy category of 4-dimensional manifolds [5] and on the secondary cohomology operations [4] is based on the “calculus of tracks”. One of the main tricks in [4] is to make some track theories strong. The aim of this and some subsequent papers is to shed more light on this procedure. In this paper we prove that certain track theories are equivalent to strong ones. To be more precise, let us fix some terminology.

A *track category* is a category enriched in groupoids. Thus it consists of objects, 1-arrows between them, and 2-arrows, or tracks, between 1-arrows with the same source and target, and for two objects  $X, Y$  of a track category  $\mathcal{T}$  there is their *Hom-groupoid*  $\llbracket X, Y \rrbracket_{\mathcal{T}}$ , or just  $\llbracket X, Y \rrbracket$ , whose objects are 1-arrows  $X \rightarrow Y$  and morphisms are 2-arrows. Objects  $X, Y$  of a track category will be called *homotopy equivalent* if there are 1-arrows  $f : X \rightarrow Y, g : Y \rightarrow X$  with the composites  $fg, gf$  isomorphic to identities. A track category is *abelian* if for any 1-arrow  $f : X \rightarrow Y$ , the group  $\text{Aut}(f)$  of tracks from  $f$  to itself is abelian.

Two track categories  $\mathcal{T}, \mathcal{T}'$  are called *weakly equivalent* if there is an enriched functor  $F : \mathcal{T} \rightarrow \mathcal{T}'$  which induces equivalences of hom-groupoids  $\llbracket X, Y \rrbracket_{\mathcal{T}} \rightarrow \llbracket FX, FY \rrbracket_{\mathcal{T}'}$  and is *essentially surjective*, i. e. any object of  $\mathcal{T}'$  is homotopy equivalent to one of the form  $FX$ .

A *track theory* for us is a track category  $\mathcal{T}$  possessing finite lax products; this means that for any objects  $X, Y$  of  $\mathcal{T}$  there is an object  $X \times Y$  with 1-arrows  $X \times Y \rightarrow X, X \times Y \rightarrow Y$  such that the induced functors between groupoids

$$\llbracket Z, X \times Y \rrbracket \rightarrow \llbracket Z, X \rrbracket \times \llbracket Z, Y \rrbracket$$

are equivalences of groupoids for all objects  $Z$ .

A track theory is *strong* if the above functors are in fact isomorphisms of groupoids.

Morphisms of track theories are enriched functors which are compatible with lax products. An equivalence of track theories is a track theory morphism which is a weak equivalence and two track theories are called *equivalent* if they are made so by the smallest equivalence relation generated by these.

Our main theorem is that any abelian track theory  $\mathcal{T}$  is equivalent to a strong one. The fact itself is a trivial consequence of our results on cohomological properties of algebraic theories. We believe there exists another, more direct proof of this, and probably more general result. However the cohomological results that we obtain are of independent interest in view of applications to topological Hochschild cohomology [13], [19], [21].

In [13] the second and third author defined the cohomology of algebraic theories with some coefficients. In the present paper we extend this definition in two directions. First, we pass from single sorted theories to multisorted ones, to obtain our main theorem in full generality. Second, we extend coefficients for cohomology. This is necessary for proving our main theorem even for the particular case of single sorted theories.

For a theory  $\mathbb{A}$  we introduce an abelian category  $\mathcal{F}(\mathbb{A})$  in such a way that the Ext groups in this category yield cohomology groups of  $\mathbb{A}$ . This category is in general bigger than the one introduced in [13], although it is the same in the important particular case when  $\mathbb{A}$  is the theory of modules over a ring — see [14]. The new cohomologies, just as the old ones, are closely related to the Baues-Wirsching cohomologies [9] of categories; moreover whereas the old coefficients correspond to the Baues-Wirsching cohomologies of categories with coefficients in bifunctors, our new extended coefficients correspond to the Baues-Wirsching cohomologies with coefficients in more general natural systems.

To get a hint of what new coefficients are, and what kind of cohomology groups can arise, let us take an example when  $\mathbb{A}$  is the theory of groups  $\mathbb{Gr}$ . The corresponding coefficient systems according to [13] were functors from the category of finitely generated free groups to the category of abelian groups. As we said, in the present paper we consider more general coefficients, they form the category  $\mathcal{F}(\mathbb{Gr})$ , which consists of assignments  $M$  of an  $F$ -module  $M_F$  to each finitely generated free group  $F$ , in a way which is functorial in  $F$ . Then coefficients in the sense of [13] correspond to those objects  $M$  of  $\mathcal{F}(\mathbb{Gr})$  for which the  $F$ -module structure on  $M_F$  is trivial for all  $F$ . One typical object of  $\mathcal{F}(\mathbb{Gr})$  is, for example,  $\Omega^1$ , which assigns to the group  $F$  the augmentation ideal  $\Omega_F^1 \subset \mathbb{Z}[F]$  of its group ring, considered as an  $F$ -submodule of  $\mathbb{Z}[F]$ . We will then have, for any other object  $M$  of  $\mathcal{F}(\mathbb{Gr})$ , the groups  $\text{Ext}_{\mathcal{F}(\mathbb{Gr})}^*(\Omega^1, M)$  which will be cohomology groups of  $\mathbb{Gr}$  with coefficients in  $M$ . We will see that an object similar to  $\Omega^1$  exists in general, and this construction will be naturally extended to any theory in place of  $\mathbb{Gr}$ .

One of our main results is that this new cohomology is trivial in dimensions  $> 1$  for free theories, just as the old one with more restricted coefficients. This result together with relationship between third cohomology group and track extensions [17], [18], [6] gives our main result on strengthening of track theories.

## 2. ABELIAN TRACK CATEGORIES AND COHOMOLOGY OF SMALL CATEGORIES

**2.1. Groupoids, tracks and track categories.** Recall that a *groupoid* is a category all of whose morphisms are invertible. We will use the following notation. For a groupoid  $\mathbf{G}$ , the set of its objects will be denoted by  $\text{Ob}(\mathbf{G})$  and the set of morphisms by  $\text{Mor}(\mathbf{G})$ . We have the canonical source and target maps

$$\text{Mor}(\mathbf{G}) \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \text{Ob}(\mathbf{G}) .$$

A groupoid is called *abelian* if the automorphism group of each object is an abelian group.

A *2-category* is a category enriched in the category of small categories. In other words a 2-category  $\mathcal{T}$  consists of a class of objects  $\text{Ob}(\mathcal{T})$ , a collection of small categories

$\llbracket A, B \rrbracket = \llbracket A, B \rrbracket_{\mathcal{T}}$  for  $A, B \in \text{Ob } \mathcal{T}$  called *hom-categories* of  $\mathcal{T}$ , identities  $1_A \in \text{Ob}(\llbracket A, A \rrbracket)$  and composition functors  $\llbracket B, C \rrbracket \times \llbracket A, B \rrbracket \rightarrow \llbracket A, C \rrbracket$  satisfying the usual equations of associativity and identity morphisms. Objects of the hom-category  $f \in \text{Ob}(\llbracket A, B \rrbracket)$  are called *1-arrows* of  $\mathcal{T}$ , while morphisms from  $\llbracket A, B \rrbracket$  are called *2-arrows*. We will use the following notation for 2-categories. If  $f : A \rightarrow B$  and  $x : B \rightarrow C$  are 1-arrows, then the composite of  $f$  and  $x$  is denoted by  $xf : A \rightarrow C$ . Notation  $\alpha : f \Rightarrow f_1$  will indicate a 2-arrow from  $f$  to  $f_1$ , with  $f, f_1 \in \text{Ob}(\llbracket A, B \rrbracket)$ ,  $A, B \in \text{Ob}(\mathcal{T})$ . For the composition of 2-arrows we use additive notation: the identity 2-arrow  $f \Rightarrow f$  of a 1-arrow  $f$  will be denoted by  $0_f$ , and for 1-arrows  $f, g, h : A \rightarrow B$  and 2-arrows  $\alpha : f \Rightarrow g$ ,  $\beta : g \Rightarrow h$ , the composite of  $\alpha$  and  $\beta$  in the category  $\llbracket A, B \rrbracket$  is denoted by  $\beta + \alpha$ .

There are several categories associated with a 2-category  $\mathcal{T}$ . The category  $\mathcal{T}_0$  has the same objects as  $\mathcal{T}$ , while morphisms in  $\mathcal{T}_0$  are 1-arrows of  $\mathcal{T}$ . The category  $\mathcal{T}_1$  has the same objects as  $\mathcal{T}_0$ . The morphisms  $A \rightarrow B$  in  $\mathcal{T}_1$  are 2-arrows  $\alpha : f \Rightarrow f_1$  where  $f, f_1 : A \rightarrow B$  are 1-arrows in  $\mathcal{T}$ . The composition in  $\mathcal{T}_1$  is given by  $(\beta : x \Rightarrow x_1)(\alpha : f \Rightarrow f_1) := (\beta\alpha : xf \Rightarrow x_1f_1)$ , where

$$\beta\alpha = \beta f_1 + x\alpha = x_1\alpha + \beta f.$$

One furthermore has the source and target functors

$$\mathcal{T}_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} \mathcal{T}_0,$$

where  $s(\alpha : f \Rightarrow f_1) = f$  and  $t(\alpha : f \Rightarrow f_1) = f_1$ , the “identity” functor  $i : \mathcal{T}_0 \rightarrow \mathcal{T}_1$  assigning to an 1-arrow  $f$  the triple  $0_f : f \Rightarrow f$ . Moreover, consider the pullback diagram

$$\begin{array}{ccc} \mathcal{T}_1 \times_{\mathcal{T}_0} \mathcal{T}_1 & \xrightarrow{p_2} & \mathcal{T}_1; \\ p_1 \downarrow & & \downarrow t \\ \mathcal{T}_1 & \xrightarrow{s} & \mathcal{T}_0 \end{array}$$

there is also the “composition” functor  $m : \mathcal{T}_1 \times_{\mathcal{T}_0} \mathcal{T}_1 \rightarrow \mathcal{T}_1$  sending  $(\alpha : f \Rightarrow f_1, \alpha' : f_2 \Rightarrow f)$  to  $\alpha + \alpha' : f_2 \Rightarrow f_1$ . Note that these functors satisfy the identities  $sp_1 = tp_2$ ,  $sm = sp_2$ ,  $tm = tp_1$  and  $si = ti = \text{id}_{\mathcal{T}_0}$ . Sometimes we will also simply write  $\mathcal{T}_1 \rightrightarrows \mathcal{T}_0$  to indicate a 2-category  $\mathcal{T}$ .

A *track category*  $\mathcal{T}$  is a *category enriched in groupoids*, i. e. is the same as a 2-category all of whose 2-arrows are invertible. If the groupoids  $\llbracket A, B \rrbracket$  are abelian for all  $A, B \in \text{Ob } \mathcal{T}$ , then  $\mathcal{T}$  is called an *abelian track category*. For track categories we might occasionally talk about *maps* instead of 1-arrows and *homotopies* or *tracks* instead of 2-arrows. If there is a homotopy  $\alpha : f \Rightarrow g$  between maps  $f, g \in \text{Ob}(\llbracket A, B \rrbracket)$ , we will say that  $f$  and  $g$  are homotopic and write  $f \simeq g$ . Since the homotopy relation is a natural equivalence relation on morphisms of  $\mathcal{T}_0$ , it determines the *homotopy category*  $\mathcal{T}_{\simeq} = \mathcal{T}_0 / \simeq$ . Objects of  $\mathcal{T}_{\simeq}$  are once again objects in  $\text{Ob}(\mathcal{T})$ , while morphisms of  $\mathcal{T}_{\simeq}$  are homotopy classes of morphisms in  $\mathcal{T}_0$ . For objects  $A$  and  $B$  we let  $[A, B]$  denote the set of morphisms from  $A$  to  $B$  in the category  $\mathcal{T}_{\simeq}$ . Thus

$$[A, B] = \llbracket A, B \rrbracket / \simeq.$$

Usually we let  $q : \mathcal{T}_0 \rightarrow \mathcal{T}_\simeq$  denote the quotient functor. Sometimes for a 1-arrow  $f$  in  $\mathcal{T}$  we will denote  $q(f)$  by  $[f]$ . A map  $f : A \rightarrow B$  is a *homotopy equivalence* if there exists a map  $g : B \rightarrow A$  and tracks  $fg \simeq 1$  and  $gf \simeq 1$ . This is the case if and only if  $q(f)$  is an isomorphism in the homotopy category  $\mathcal{T}_\simeq$ . In this case  $A$  and  $B$  are called *homotopy equivalent* objects.

A *track functor*  $F : \mathcal{T} \rightarrow \mathcal{T}'$  between track categories is a groupoid enriched functor. Thus  $F$  assigns to each  $A \in \text{Ob}(\mathcal{T})$  an object  $F(A) \in \text{Ob}(\mathcal{T}')$ , to each map  $f : A \rightarrow B$  in  $\mathcal{T}$  — a map  $F(f) : F(A) \rightarrow F(B)$ , and to each track  $\alpha : f \Rightarrow g$  for  $f, g : A \rightarrow B$ , a track  $F(\alpha) : F(f) \Rightarrow F(g)$  in a functorial way, i. e. so that one gets functors

$$F_{A,B} : \llbracket A, B \rrbracket_{\mathcal{T}} \rightarrow \llbracket F(A), F(B) \rrbracket_{\mathcal{T}'}.$$

Moreover these assignments are compatible with identities and composition, or equivalently induce a functor  $\mathcal{T}_1 \rightarrow \mathcal{T}'_1$ , that is,  $F(1_A) = 1_{F(A)}$  for  $A \in \text{Ob}(\mathcal{T})$ ,  $F(fg) = F(f)F(g)$ , and  $F(\alpha\beta) = F(\alpha)F(\beta)$  for any  $\alpha : f \Rightarrow f_1$ ,  $\beta : g \Rightarrow g_1$ ,  $f, f_1 : B \rightarrow C$ ,  $g, g_1 : A \rightarrow B$  in  $\mathcal{T}$ .

A track functor  $F : \mathcal{T} \rightarrow \mathcal{T}'$  is called a *weak equivalence* between track categories if the functors  $\llbracket A, B \rrbracket \rightarrow \llbracket F(A), F(B) \rrbracket$  are equivalences of groupoids for all objects  $A, B$  of  $\mathcal{T}$ , and each object  $A'$  of  $\mathcal{T}'$  is homotopy equivalent to some object of the form  $F(A)$ . Such a weak equivalence induces a functor  $F : \mathcal{T}_\simeq \rightarrow \mathcal{T}'_\simeq$  between homotopy categories which is an equivalence of categories.

**2.2. Preliminaries on cohomology of small categories.** For us is a crucial fact that any abelian track category defines an element in the third cohomology group of the corresponding homotopy category with coefficients in a natural system [17], [18], [6]. Therefore we recall the corresponding notions.

Let  $\mathbf{C}$  be a category. Then the category  $\mathbf{FC}$  of factorizations in  $\mathbf{C}$  is defined as follows. Objects of  $\mathbf{FC}$  are morphisms  $f : A \rightarrow B$  in  $\mathbf{C}$  and morphisms  $(a, b) : f \rightarrow g$  in  $\mathbf{FC}$  are commutative diagrams

$$\begin{array}{ccc} A & \xleftarrow{a} & A' \\ \downarrow f & & \downarrow g \\ B & \xrightarrow{b} & B' \end{array}$$

in the category  $\mathbf{C}$ . A *natural system* on  $\mathbf{C}$  is a functor  $D : \mathbf{FC} \rightarrow \mathcal{A}$  to the category of abelian groups. We write  $D(f) = D_f$ . If  $a : C \rightarrow D$ ,  $f : A \rightarrow C$  and  $g : D \rightarrow B$  are morphisms in  $\mathbf{C}$ , then the induced homomorphism  $(1_A, a)_* : D_f \rightarrow D_{af}$  will be denoted by  $\xi \mapsto a\xi$ , for  $\xi \in D_f$ , while  $(a, 1_B)_* : D_g \rightarrow D_{ga}$  will be denoted by  $\eta \mapsto \eta a$ ,  $\eta \in D_g$ . We denote by  $C^*(\mathbf{C}; D)$  the following cochain complex:

$$C^m(\mathbf{C}; D) = \prod_{\left( A_0 \xleftarrow{a_1} A_1 \xleftarrow{\dots} \xleftarrow{a_n} A_n \right) \in \mathbf{C}} D_{a_1 \dots a_n},$$

with the coboundary map given by

$$\begin{aligned} d(\varphi)(a_1, a_2, \dots, a_{n+1}) &= a_1\varphi(a_2, \dots, a_{n+1}) + \\ &+ \sum_{i=1}^n (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} \varphi(a_1, \dots, a_n) a_{n+1}. \end{aligned}$$

According to [9] the cohomology  $H^*(\mathbf{C}; D)$  of  $\mathbf{C}$  with coefficients in  $D$  is defined as the homology of the cochain complex  $C^*(\mathbf{C}; D)$ .

A morphism of natural systems is just a natural transformation. For a functor  $q : \mathbf{C}' \rightarrow \mathbf{C}$ , any natural system  $D$  on  $\mathbf{C}$  gives a natural system  $D \circ (\mathbf{F}q)$  on  $\mathbf{C}'$  which we will denote  $q^*(D)$ . There is a canonical functor  $\mathbf{F}\mathbf{C} \rightarrow \mathbf{C}^{\text{op}} \times \mathbf{C}$  which assigns the pair  $(A, B)$  to  $f : A \rightarrow B$ . This functor allows one to consider any bifunctor  $D : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathcal{M}$  as a natural system. In what follows bifunctors are considered as natural systems via this correspondence. Similarly, one has a projection  $\mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{C}$ , which yields the functor  $\mathbf{F}\mathbf{C} \rightarrow \mathbf{C}$  given by  $(a : A \rightarrow B) \mapsto B$ . This allows us to consider any functor on  $\mathbf{C}$  as a natural system on  $\mathbf{C}$ . In particular one can talk about cohomology of a category  $\mathbf{C}$  with coefficients in bifunctors and in functors as well. One easily sees that for a bifunctor  $D : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathcal{M}$  the group  $H^0(\mathbf{C}; D)$  coincides with the *end* of the bifunctor  $D$  (see [15]), which consists of all families  $(x_C)_{C \in \text{Ob } \mathbf{C}}$ , where  $x_C \in D_{1_C}$ , for each  $C \in \text{Ob } \mathbf{C}$ , satisfying the condition  $a(x_A) = (x_B)a$  for all  $a : A \rightarrow B$ . In the case of a functor  $F : \mathbf{C} \rightarrow \mathcal{M}$  the group  $H^0(\mathbf{C}; F)$  is isomorphic to the limit of the functor  $F$  and the groups  $H^*(\mathbf{C}; F)$  are isomorphic to the higher limits (see [9]).

**Lemma 2.2.1.** *Let  $\mathbf{C}$  be a small category with an initial object  $i$ . Then for any functor  $F : \mathbf{C} \rightarrow \mathcal{M}$ , one has*

$$\begin{aligned} H^0(\mathbf{C}; F) &\cong F(i), \\ H^n(\mathbf{C}; F) &= 0 \text{ for } n > 0. \end{aligned}$$

*Proof.* In this case, the evaluation of a functor  $F$  at  $i$  is isomorphic to the limit of  $F$ . Thus  $\lim$  is an exact functor and therefore higher limits vanish.  $\square$

*Example 2.2.2.* Let  $F, G : \mathbf{C} \rightarrow R\text{-mod}$  be two functors to the category of left  $R$ -modules, for a ring  $R$ . One can define the bifunctor  $\mathcal{H}om(F, G) : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathcal{M}$  by

$$\mathcal{H}om(F, G)(C, D) := \text{Hom}_R(F(C), G(D)).$$

Then  $H^0(\mathbf{C}; \mathcal{H}om(F, G)) \cong \text{Hom}_{\mathcal{A}}(F, G)$ , where  $\mathcal{A}$  is the category of all functors from  $\mathbf{C}$  to  $R\text{-mod}$ . Moreover, if  $F(C)$  is a projective  $R$ -module for all  $C \in \text{Ob } \mathbf{C}$ , then there is an isomorphism  $H^*(\mathbf{C}; \mathcal{H}om(F, G)) \cong \text{Ext}_{\mathcal{A}}^*(F, G)$  [13]. We will need a generalization of these facts, when  $R$  is not a constant ring, but a functor from  $\mathbf{C}$  to the category of rings — see 3 below. In this case it is necessary to switch to natural systems instead of bifunctors.

Note that for any  $\mathbf{C}$  there is a canonical isomorphism of categories  $\mathbf{F}\mathbf{C} \cong \mathbf{F}(\mathbf{C}^{\text{op}})$ , which is identity on objects. Using this we will identify natural systems on  $\mathbf{C}$  and on  $\mathbf{C}^{\text{op}}$  everywhere in the sequel.

**2.3. Track extensions and third cohomology of small categories.** As was discovered in [7] if a track category  $\mathcal{T}$  is abelian, then one has an additional structure. To describe it we need more notions [9], [6].

Let  $\mathcal{B}$  be a 2-category. There is a natural system  $\text{End}_{\mathcal{B}}$  of monoids on  $\mathcal{B}_0$  (i. e. a functor  $\mathbf{F}\mathcal{B}_0 \rightarrow \text{Monoids}$ ) which assigns to an 1-arrow  $f : A \rightarrow B$  the monoid of all 2-arrows  $f \Rightarrow f$  in  $\mathcal{B}$ . Indeed for  $g : B \rightarrow B'$ ,  $h : A' \rightarrow A$  morphisms in  $\mathcal{B}_0$  we already defined the induced homomorphisms:

$$\begin{aligned} (\varepsilon \mapsto g\varepsilon) : \text{Hom}_{\mathcal{B}}(f, f) &\rightarrow \text{Hom}_{\mathcal{B}}(gf, gf), \\ (\varepsilon \mapsto \varepsilon h) : \text{Hom}_{\mathcal{B}}(f, f) &\rightarrow \text{Hom}_{\mathcal{B}}(fh, fh). \end{aligned}$$

For a track category  $\mathcal{T}$ , clearly  $\text{End}_{\mathcal{T}} = \text{Aut}_{\mathcal{T}}$  takes values in the category of groups. It turns out that the natural system  $\text{Aut}_{\mathcal{T}}$  has an additional structure. To describe it let us introduce the following definition.

**Definition 2.3.1.** Consider a track category  $\mathcal{T}$ . Since taking the category of factorizations from 2.2 is evidently functorial, applying it to constituents of  $\mathcal{T}$  gives the diagram

$$\mathbf{F}(\mathcal{T}_1 \times_{\mathcal{T}_0} \mathcal{T}_1) \begin{array}{c} \xrightarrow{\mathbf{F}p_1} \\ \xrightarrow{\mathbf{F}m} \\ \xrightarrow{\mathbf{F}p_2} \end{array} \mathbf{F}\mathcal{T}_1 \begin{array}{c} \xleftarrow{\mathbf{F}s} \\ \xleftarrow{\mathbf{F}i} \\ \xleftarrow{\mathbf{F}t} \end{array} \mathbf{F}\mathcal{T}_0,$$

where the functors  $p_1, m, p_2, s, t, i$  are as in 2.1.

A  $\mathcal{T}$ -natural system with values in a category  $\mathcal{C}$  is a natural system  $D : \mathbf{F}\mathcal{T}_0 \rightarrow \mathcal{C}$  on  $\mathcal{T}_0$  together with a natural transformation  $\nabla : D \circ \mathbf{F}s \rightarrow D \circ \mathbf{F}t$  such that the diagrams

$$\begin{array}{ccc} & D & \\ \swarrow & & \searrow \\ D \circ \mathbf{F}s \circ \mathbf{F}i & \xrightarrow{\nabla \mathbf{F}i} & D \circ \mathbf{F}t \circ \mathbf{F}i \end{array}$$

and

$$\begin{array}{ccccc} & D \circ \mathbf{F}s \circ \mathbf{F}p_1 & \xlongequal{\quad} & D \circ \mathbf{F}t \circ \mathbf{F}p_2 & \\ \swarrow \nabla \mathbf{F}p_1 & & & & \nwarrow \nabla \mathbf{F}p_2 \\ D \circ \mathbf{F}t \circ \mathbf{F}p_1 & & & & D \circ \mathbf{F}s \circ \mathbf{F}p_2 \\ & \searrow & & \swarrow & \\ & D \circ \mathbf{F}t \circ \mathbf{F}m & \xleftarrow{\nabla \mathbf{F}m} & D \circ \mathbf{F}s \circ \mathbf{F}m & \end{array}$$

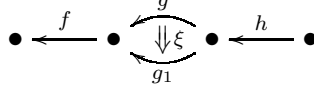
commute.

Unfolding this definition in terms of elements one sees easily that a  $\mathcal{T}$ -natural system is the same as a natural system  $D$  together with a family of morphisms

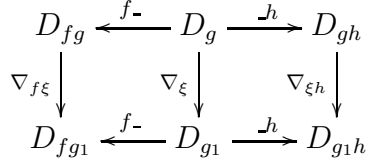
$$\nabla_{\xi} : D_f \rightarrow D_g$$

in the category  $\mathcal{C}$ , one for each track  $\xi : f \Rightarrow g$  in  $\mathcal{T}$ , such that the following conditions are satisfied:

- i)  $\nabla_{0_f} = \text{id}_{D_f}$  for all 1-arrows  $f$  in  $\mathcal{T}$ .
- ii) For  $\xi : f \Rightarrow g, \eta : g \Rightarrow h$  one has  $\nabla_{\eta+\xi} = \nabla_\eta \circ \nabla_\xi$ .
- iii) For a diagram

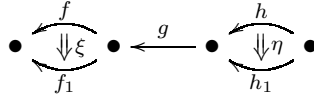


the following diagram

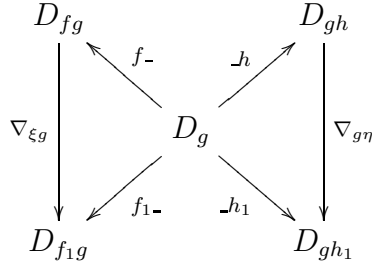


commutes.

- iv) For a diagram



the diagram



commutes.

A morphism  $\Phi : (D, \nabla) \rightarrow (D', \nabla')$  of  $\mathcal{T}$ -natural systems is a natural transformation  $\Phi$  between the functors  $D, D' : \mathbf{F}\mathcal{T}_0 \rightarrow \mathcal{C}$ , such that the diagram

$$\begin{array}{ccc} D \circ \mathbf{F}s & \xrightarrow{\Phi \mathbf{F}s} & D' \circ \mathbf{F}s \\ \nabla \downarrow & & \downarrow \nabla' \\ D \circ \mathbf{F}t & \xrightarrow{\Phi \mathbf{F}t} & D' \circ \mathbf{F}t \end{array}$$

commutes. We denote by  $\mathcal{T}\text{-Nat}$  the category of  $\mathcal{T}$ -natural systems.

Let  $G : \mathcal{T}' \rightarrow \mathcal{T}$  be a track functor. For any  $\mathcal{T}$ -natural system  $(D, \nabla)$  one defines a  $\mathcal{T}'$ -natural system  $G^*(D, \nabla) = (D \circ \mathbf{F}G, \nabla G)$ , where for  $\xi' : f' \Rightarrow g'$  in  $\mathcal{T}'$ ,  $(\nabla G)_{\xi'} : D_{Gf'} \rightarrow D_{Gg'}$  is defined to be  $\nabla_{G\xi'}$ . In this way one obtains a functor

$$G^* : \mathcal{T}\text{-Nat} \rightarrow \mathcal{T}'\text{-Nat}.$$

*Example 2.3.2.* For a track category  $\mathcal{T}$ , the group-valued natural system  $\text{Aut}_{\mathcal{T}}$  is equipped with a canonical structure of a  $\mathcal{T}$ -natural system given by

$$\nabla_{\xi}(a) = \xi + a - \xi.$$

Let  $D$  be a natural system on  $\mathcal{T}_{\simeq}$ . Then  $q^*D$  is a natural system on  $\mathcal{T}_0$  given by  $(q^*D)_f = D_{q(f)}$ . Here  $q : \mathcal{T}_0 \rightarrow \mathcal{T}_{\simeq}$  is the canonical projection. Define the structure of a  $\mathcal{T}$ -natural system on  $q^*D$  by  $\nabla = \text{id} : D \circ \mathbf{F}q \circ \mathbf{F}s = D \circ \mathbf{F}q \circ \mathbf{F}t$ . In this way one obtains the functor  $q^* : \text{Nat}(\mathcal{T}_{\simeq}) \rightarrow \mathcal{T}\text{-Nat}$ . Our Theorem 2.3.3 claims that the functor  $q^*$  is a full embedding. Actually we also identify the essential image of the functor  $q^*$ . We need the following definition. A  $\mathcal{T}$ -natural system  $(D, \nabla)$  is called *inert* if  $\nabla_{\varepsilon} = \text{id}_f$  for all  $\varepsilon : f \Rightarrow f$ . Inert  $\mathcal{T}$ -natural systems form a full subcategory of the category of  $\mathcal{T}$ -natural systems, which is denoted by  $\mathcal{T}\text{-Inert}$ . It is clear that the image of the functor  $q^*$  lies in  $\mathcal{T}\text{-Inert}$ . It is also clear that  $\text{Aut}_{\mathcal{T}}$  equipped with the canonical  $\mathcal{T}$ -natural system structure defined in Example 2.3.2 is inert if and only if  $\mathcal{T}$  is an abelian track category.

Let us observe that for any track functor  $G : \mathcal{T}' \rightarrow \mathcal{T}$  restriction of the functor  $G^* : \mathcal{T}\text{-Nat} \rightarrow \mathcal{T}'\text{-Nat}$  yields the functor  $G^* : \mathcal{T}\text{-Inert} \rightarrow \mathcal{T}'\text{-Inert}$ .

**Theorem 2.3.3.** *Let  $\mathcal{T}$  be a track category. Then  $q^* : \text{Nat}(\mathcal{T}_{\simeq}) \rightarrow \mathcal{T}\text{-Inert}$  is an equivalence of categories. Furthermore, for any track functor  $G : \mathcal{T}' \rightarrow \mathcal{T}$  the diagram*

$$\begin{array}{ccc} \text{Nat}(\mathcal{T}_{\simeq}) & \xrightarrow{q^*} & \mathcal{T}\text{-Inert} \\ \downarrow G_{\simeq}^* & & \downarrow G^* \\ \text{Nat}(\mathcal{T}'_{\simeq}) & \xrightarrow{q'^*} & \mathcal{T}'\text{-Inert} \end{array}$$

*commutes.*

*Proof.* Let  $E$  and  $E'$  be natural systems on  $\mathcal{T}_{\simeq}$  and let  $\Phi : q^*E \rightarrow q^*E'$  be a morphism of  $\mathcal{T}$ -natural systems. We claim that if  $f$  and  $g$  are homotopic maps in  $\mathcal{T}_0$  (and therefore  $qf = qg$ ), then the homomorphisms  $\Phi_f : E_{qf} \rightarrow E'_{qf}$  and  $\Phi_g : E_{qg} \rightarrow E'_{qg}$  are the same. Indeed, we can choose a track  $\xi : f \Rightarrow g$ . Then we have the following commutative diagram:

$$\begin{array}{ccc} (q^*E)_f & \xrightarrow{\nabla_{\xi}} & (q^*E)_g \\ \Phi_f \downarrow & & \downarrow \Phi_g \\ (q^*E')_f & \xrightarrow{\nabla'_{\xi}} & (q^*E')_g \end{array}$$

By definition of the  $\mathcal{T}$ -natural system structure on  $q^*E$  and  $q^*E'$  the morphisms  $\nabla_{\xi}$  and  $\nabla'_{\xi}$  are the identity morphisms, hence the claim. This shows that the functor  $q^*$  is full and faithful.

It remains to show that for any inert  $\mathcal{T}$ -natural system  $(D, \nabla)$  there exists a natural system  $E$  on  $\mathcal{T}_{\simeq}$  and an isomorphism  $\Delta : D \rightarrow q^*E$  of  $\mathcal{T}$ -natural systems. First of all one observes that if  $\xi, \eta : f \Rightarrow g$  are tracks, then  $\nabla_{\xi} = \nabla_{\eta} : D_f \rightarrow D_g$ . Indeed, thanks to the property ii) of Definition 2.3.1 we have

$$\nabla_{\xi} = \nabla_{\xi - \eta + \eta} = \nabla_{\xi - \eta} \nabla_{\eta} = \nabla_{\eta},$$



because  $\xi - \eta : g \Rightarrow g$  and  $D$  is inert. Therefore for  $qf = qg$  there is a well defined homomorphism  $\nabla_{f,g} : D_f \rightarrow D_g$  induced by any track  $f \Rightarrow g$ . Then the relation ii) of Definition 2.3.1 shows that  $\nabla_{g,h}\nabla_{f,g} = \nabla_{f,h}$  for any composable 1-arrows  $f, g, h$ . By harmless abuse of notation we will just write  $\nabla$  instead of  $\nabla_{f,g}$  in what follows.

Since the functor  $q : \mathcal{T}_0 \rightarrow \mathcal{T}_\simeq$  is identity on objects and full, we can choose for any arrow  $a$  in  $\mathcal{T}_\simeq$  a map  $u(a)$  in  $\mathcal{T}_0$  such that  $qu(a) = a$ . Moreover for any map  $f$  in  $\mathcal{T}_0$  we can choose a track  $\delta(f) : f \Rightarrow u(qf)$ . Now we put

$$E_a := D_{u(a)} \text{ and } \Delta_f := \nabla = \nabla_{f, u(qf)} = \nabla_{\delta(f)} : D_f \rightarrow D_{u(qf)} = E_{qf}.$$

For a diagram  $\xleftarrow{c} \xleftarrow{a} \xleftarrow{b}$  in the category  $\mathcal{T}_\simeq$  we define the homomorphism  $c_- : E_a \rightarrow E_{ca}$  to be the following composite:

$$E_a = D_{u(a)} \xrightarrow{u(c)_-} D_{u(c)u(a)} \xrightarrow{\nabla} D_{u(ca)} = E_{ca}.$$

Similarly we define the homomorphisms  $_b : E_a \rightarrow E_{ab}$  to be the following composites:

$$E_a = D_{u(a)} \xrightarrow{u(b)_-} D_{u(a)u(b)} \xrightarrow{\nabla} D_{u(ab)} = E_{ab}.$$

It follows from the property iii) of Definition 2.3.1 that for any diagram  $\xleftarrow{c_1} \xleftarrow{c} \xleftarrow{a}$  in the category  $\mathcal{T}_\simeq$  we have the following commutative diagram:

$$\begin{array}{ccccc} D_{u(a)} & & & & \\ u(c)_- \downarrow & \searrow c_- & & & \\ D_{u(c)u(a)} & \xrightarrow{\nabla} & D_{u(ca)} & & \\ u(c_1)_- \downarrow & & u(c_1)_- \downarrow & \searrow c_{1-} & \\ D_{u(c_1)u(c)u(a)} & \xrightarrow{\nabla} & D_{u(c_1)u(ca)} & \xrightarrow{\nabla} & D_{u(c_1ca)} \end{array}$$

Thus  $c_1(c_-) = \nabla(u(c_1)(u(c)_-))$ . On the other hand by definition we have the commutative diagram:

$$\begin{array}{ccc} D_{u(a)} & & \\ u(c_1c)_- \downarrow & \searrow (c_1c)_- & \\ D_{u(c_1c)u(a)} & \xrightarrow{\nabla} & D_{u(c_1ca)} \end{array}$$

It follows from the property iv) of Definition 2.3.1 that one has also the following commutative diagram

$$\begin{array}{ccc} D_{u(a)} & \xrightarrow{(u(c_1)u(c))_-} & D_{u(c_1)u(c)u(a)} \\ u(c_1c)_- \downarrow & \swarrow \nabla & \nabla \downarrow \\ D_{u(c_1c)u(a)} & \xrightarrow{\nabla} & D_{u(c_1ca)} \end{array}$$

Therefore

$$(c_1c)_- = \nabla(u(c_1c)_-) = \nabla(\nabla(u(c_1)(u(c)_-))) = (c_1(c_-)).$$

Similarly  $\lrcorner(b_1b) = (\lrcorner b_1)b$  and  $E$  is a well-defined natural system on  $\mathcal{T}_\simeq$ . It remains to show that  $\Delta : D \rightarrow q^*E$  is a natural transformation of functors defined on  $\mathbf{F}\mathcal{T}_0$ . To this end, one observes that for any composable morphisms  $g, f$  in the category  $\mathcal{T}_0$  we have the following commutative diagram

$$\begin{array}{ccccccc}
 D_f & \xrightarrow{\nabla} & D_{u(qf)} & & & & \\
 \downarrow g- & & \downarrow g- & \searrow u(qg)- & & & \\
 D_{gf} & \xrightarrow{\nabla} & D_{gu(qf)} & \xrightarrow{\nabla} & D_{u(qg)u(qf)} & \xrightarrow{\nabla} & D_{uq(gf)} \\
 & & & \searrow \nabla & & & 
 \end{array}$$

This means that the following diagram also commutes:

$$\begin{array}{ccc}
 D_f & \xrightarrow{\Delta_f} & E_{qf} \\
 \downarrow g- & & \downarrow (qg)- \\
 D_{gf} & \xrightarrow{\Delta_{gf}} & E_{q(gf)}
 \end{array}$$

Similarly the diagram

$$\begin{array}{ccc}
 D_g & \xrightarrow{\Delta_g} & E_{qg} \\
 \downarrow \lrcorner f & & \downarrow \lrcorner(qf) \\
 D_{gf} & \xrightarrow{\Delta_{gf}} & E_{q(gf)}
 \end{array}$$

also commutes and therefore  $\Delta$  is indeed a natural transformation.  $\square$

Now let  $\mathcal{T}$  be an abelian track category, so that  $\text{Aut}_{\mathcal{T}}$  is a natural system on  $\mathcal{T}_0$  with values in the category of abelian groups. According to Example 2.3.2 it is equipped with the canonical structure of a  $\mathcal{T}$ -natural system, which is moreover inert, because  $\mathcal{T}$  is abelian. Thus one can use Theorem 2.3.3 to conclude that there is a natural system  $D$  defined on  $\mathcal{T}_\simeq$  and an isomorphism of  $\mathcal{T}$ -natural systems  $\tau : \text{Aut}_{\mathcal{T}} \rightarrow q^*D$  defined on  $\mathcal{T}_0$ . Roughly speaking a linear track extension is a choice of such an isomorphism, which is unique up to a unique isomorphism, in the following sense: if  $(D_1, \tau_1)$  is another pair satisfying the same property, then thanks to Theorem 2.3.3 there is a unique isomorphism  $\sigma : D \rightarrow D_1$  making the following diagram commute:

$$\begin{array}{ccc}
 q^*D & \xrightarrow{q^*\sigma} & q^*D_1 \\
 \searrow \tau & & \swarrow \tau_1 \\
 & \text{Aut}_{\mathcal{T}} & 
 \end{array}$$

**Definition 2.3.4.** Let  $\mathbf{C}$  be a small category and let  $D : \mathbf{FC} \rightarrow \mathcal{M}$  be a natural system on  $\mathbf{C}$ . A linear track extension of  $\mathbf{C}$  by  $D$  denoted by

$$0 \rightarrow D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \rightarrow \mathbf{C} \rightarrow 0$$

is a pair  $(\mathcal{T}, \tau)$ . Here  $\mathcal{T}$  is an abelian track category equipped with a functor  $q : \mathcal{T}_0 \rightarrow \mathbf{C}$  which is full and identity on objects. In addition for maps  $f, g$  in  $\mathcal{T}_0$  we have  $q(f) = q(g)$  iff  $f \simeq g$ . In other words the functor  $q$  identifies  $\mathbf{C}$  with  $\mathcal{T}_\simeq$ . Furthermore  $\tau : q^*D \rightarrow \text{Aut}_{\mathcal{T}}$  is an isomorphism of  $\mathcal{T}$ -natural systems, where  $\text{Aut}_{\mathcal{T}}$  is considered as a  $\mathcal{T}$ -natural systems as in Example 2.3.2.

Hence by virtue of 2.3.3 any abelian track category  $\mathcal{T}$  is a part of the linear track extension

$$0 \rightarrow D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \rightarrow \mathcal{T}_\simeq \rightarrow 0,$$

with a natural system  $D$ , which is defined uniquely up to a canonical isomorphism.

Let  $\mathbf{C}$  be a small category and let  $D$  be a natural system on  $\mathbf{C}$ . Objects of the category  $\text{Trext}(\mathbf{C}; D)$  are linear track extensions

$$0 \rightarrow D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \xrightarrow{q} \mathbf{C} \rightarrow 0$$

and morphisms are track functors  $F : \mathcal{T} \rightarrow \mathcal{T}'$  for which  $q'F = q$  and  $\sigma' = F\sigma$ .

**Lemma 2.3.5.** *Any morphism between track extensions of a small category  $\mathbf{C}$  by a natural system  $D$  is a weak equivalence.*

*Proof.* Consider one such morphism represented by  $F : \mathcal{T} \rightarrow \mathcal{T}'$ . First of all, since  $F$  is identity on objects, every object of  $\mathcal{T}'$  is equivalent — in fact, equal — to an object of the form  $FX$ . Consider now the induced functors between Hom-groupoids  $F_{X,Y} : \llbracket X, Y \rrbracket_{\mathcal{T}} \rightarrow \llbracket FX, FY \rrbracket_{\mathcal{T}'}$ . These functors are essentially surjective on objects since  $\mathcal{T}_\simeq = \mathcal{T}'_\simeq = \mathbf{C}$  implies that for any  $f' : X \rightarrow Y$  in  $\mathcal{T}'$  there is an  $f : X \rightarrow Y$  in  $\mathcal{T}$  with  $qf = q'f'$ . Then  $q'Ff = qf = q'f'$  implies  $Ff$  and  $f'$  must be homotopic. Next the  $F_{X,Y}$  are all full since  $q'F = q$  implies that whenever  $Ff$  and  $Fg$  are homotopic,  $f$  and  $g$  must be homotopic too, for any  $f, g : X \rightarrow Y$  in  $\mathcal{T}$ . Finally  $\sigma' = F\sigma$  implies that the group homomorphisms  $\text{Aut}_{\mathcal{T}}(f) \rightarrow \text{Aut}_{\mathcal{T}'}(Ff)$  are all isomorphisms. This then clearly implies that all the  $F_{X,Y}$  are equivalences of groupoids.  $\square$

**Theorem 2.3.6.** ([17], [6]) *There is a natural bijection*

$$H^3(\mathbf{C}; D) \approx \pi_0(\text{Trext}(\mathbf{C}; D)).$$

$\square$

Here and in what follows  $\pi_0$  denotes the set of connected components of a category.

Let

$$0 \rightarrow D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \xrightarrow{p} \mathbf{C} \rightarrow 0$$

be a linear track extension of  $\mathbf{C}$  by  $D$  and let  $f : \mathbf{C}' \rightarrow \mathbf{C}$  be a functor. Then one can pull back the track extension to get a linear track extension

$$0 \rightarrow D \rightarrow \mathcal{T}'_1 \rightrightarrows \mathcal{T}'_0 \xrightarrow{p'} \mathbf{C}' \rightarrow 0$$

of  $\mathbf{C}'$ . We define the track category  $f^*\mathcal{T} = \mathcal{T}'$ , as follows. The objects of  $\mathcal{T}'$  are the same as those of  $\mathbf{C}'$ . We will denote them by  $A', B'$  etc. Maps  $A' \rightarrow B'$  in  $\mathcal{T}'$  are pairs  $(x, \alpha)$ , where  $x : f(A') \rightarrow f(B')$  is a morphism in  $\mathcal{T}_0$ , while  $\alpha : A' \rightarrow B'$  is a morphism in  $\mathbf{C}'$  such

that  $p(x) = f(\alpha)$ . If  $(x, \alpha), (y, \beta)$  are maps  $A' \rightarrow B'$  in  $\mathcal{T}'$ , then tracks  $(x, \alpha) \Rightarrow (y, \beta)$  in  $\mathcal{T}'$  exist iff  $\alpha = \beta$  in  $\mathbf{C}'$ . If this condition holds then we put

$$\mathrm{Hom}_{[[A', B']]_{\mathcal{T}'}}((x, \alpha), (y, \beta)) = \mathrm{Hom}_{[[A, B]]_{\mathcal{T}}}(x, y),$$

where  $A = f(A')$  and  $B = f(B')$ . Since the underlying category  $\mathcal{T}'_0$  is the pullback of  $\mathcal{T}_0 \rightarrow \mathbf{C}$  along the functor  $f : \mathbf{C}' \rightarrow \mathbf{C}$ , we will call this construction a pullback construction. It is clear that one gets a linear track extension

$$0 \rightarrow f^*D \rightarrow \mathcal{T}'_1 \rightrightarrows \mathcal{T}'_0 \xrightarrow{p'} \mathbf{C}' \rightarrow 0,$$

where  $p'$  is identity on objects and on morphisms is given by  $p'(x, \alpha) = \alpha$ . In particular we get the map

$$f^* : \pi_0(\mathrm{Trext}(\mathbf{C}; D)) \rightarrow \pi_0(\mathrm{Trext}(\mathbf{C}'; f^*D)); \quad \mathcal{T} \mapsto f^*\mathcal{T}.$$

One easily checks that in this way we really get a linear track extension which corresponds to the map  $f^* : H^3(\mathbf{C}; D) \rightarrow H^3(\mathbf{C}'; f^*D)$ .

The proof of Theorem 2.3.6 given in [18] is based on the following Theorem 2.3.7, which is going to be crucial in this paper as well.

Let  $p : \mathbf{K} \rightarrow \mathbf{C}$  be a full functor which is identity on objects. Let  $D : \mathbf{FC} \rightarrow \mathcal{M}$  be a natural system on  $\mathbf{C}$ . We denote by  $\mathrm{Trext}(\mathbf{C}, \mathbf{K}; D)$  the subcategory of  $\mathrm{Trext}(\mathbf{C}; D)$  whose objects are track categories  $\mathcal{T}$  with  $\mathcal{T}_0 = \mathbf{K}$ . Morphisms in  $\mathrm{Trext}(\mathbf{C}, \mathbf{K}; D)$  are track functors  $\mathcal{T} \rightarrow \mathcal{T}'$  which are identity on arrows and hence induce the identity functor  $\mathbf{K} = \mathcal{T}_0 \rightarrow \mathcal{T}'_0 = \mathbf{K}$ . It is clear that  $\mathrm{Trext}(\mathbf{C}, \mathbf{K}; D)$  is a groupoid. In order to relate the set of components  $\pi_0(\mathrm{Trext}(\mathbf{C}, \mathbf{K}; D))$  of  $\mathrm{Trext}(\mathbf{C}, \mathbf{K}; D)$  to cohomology of small categories we need the following variant of the relative cohomology groups. In the above circumstances  $p^*(D)$  is a natural system on  $\mathbf{K}$ , which we will denote still by  $D$ . Then  $p$  yields a monomorphism of cochain complexes  $C^*(\mathbf{C}; D) \rightarrow C^*(\mathbf{K}; D)$ . We let  $C^*(\mathbf{C}, \mathbf{K}; D)$  be the cokernel of this homomorphism. The  $n$ -th relative cohomology group  $H^n(\mathbf{C}, \mathbf{K}; D)$  is defined as the  $(n - 1)$ -st homology group of the cochain complex  $C^*(\mathbf{C}, \mathbf{K}; D)$ . Then one has an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\mathbf{C}; D) \rightarrow H^0(\mathbf{K}; D) \rightarrow H^1(\mathbf{C}, \mathbf{K}; D) \rightarrow \dots \\ \rightarrow H^n(\mathbf{C}; D) \rightarrow H^n(\mathbf{K}; D) \rightarrow H^{n+1}(\mathbf{C}, \mathbf{K}; D) \rightarrow \dots \end{aligned}$$

**Theorem 2.3.7.** *Let  $p : \mathbf{K} \rightarrow \mathbf{C}$  be a full functor which is identity on objects and let  $D : \mathbf{FC} \rightarrow \mathcal{M}$  be a natural system. Then there is a natural bijection*

$$\pi_0(\mathrm{Trext}(\mathbf{C}, \mathbf{K}; D)) \approx H^3(\mathbf{C}, \mathbf{K}; D).$$

*Proof.* This is exactly Proposition 3.4 of [18]. □

Let

$$0 \rightarrow D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \xrightarrow{p} \mathbf{C} \rightarrow 0$$

be a linear track extension of  $\mathbf{C}$  by  $D$ . According to Theorem 2.3.6 and Theorem 2.3.7 it defines two elements  $\mathrm{Ch}(\mathcal{T}) \in H^3(\mathbf{C}; D)$  and  $\mathrm{ch}(\mathcal{T}) \in H^3(\mathbf{C}, \mathcal{T}_0; D)$ . It follows from the

proof given in [18] that  $\partial(\text{ch}(\mathcal{T})) = \text{Ch}(\mathcal{T})$ , where  $\partial : H^3(\mathbf{C}, \mathcal{T}_0; D) \rightarrow H^3(\mathbf{C}; D)$  is the connecting homomorphism in the above exact sequence.

**2.4. Lax functors and track extensions.** If two objects  $\mathcal{T}$  and  $\mathcal{T}'$  of the category  $\text{Trest}(\mathbf{C}; D)$  lie in the same connected component there is no morphism  $\mathcal{T} \rightarrow \mathcal{T}'$  in  $\text{Trest}(\mathbf{C}; D)$  in general, but only a diagram of the form  $\mathcal{T} \leftarrow \mathcal{T}'' \rightarrow \mathcal{T}'$ , with an object  $\mathcal{T}'' \in \text{Trest}(\mathbf{C}; D)$ . The aim of this section is to show that in the same circumstances there is always a lax functor from  $\mathcal{T}$  to  $\mathcal{T}'$ .

A lax functor  $F$  between 2-categories  $\mathcal{T} \rightarrow \mathcal{T}'$  consists of a map of objects  $F : \text{Ob}(\mathcal{T}) \rightarrow \text{Ob}(\mathcal{T}')$ , a collection of functors  $F_{X,Y} : \llbracket X, Y \rrbracket \rightarrow \llbracket FX, FY \rrbracket$ , for  $X, Y \in \text{Ob}(\mathcal{T})$ , a family of 2-arrows  $o_X : \text{id}_{FX} \Rightarrow F(\text{id}_X)$  for each  $X \in \text{Ob}(\mathcal{T})$ , and a natural family of 2-arrows  $a_{f,g} : (Ff)(Fg) \Rightarrow F(fg)$  for each composable pair of 1-arrows  $(f, g)$  in  $\mathcal{T}$ . These are required to satisfy *coherence conditions* — the following diagrams

$$\begin{array}{ccc} & (F \text{id}_Y)(Ff) & \\ a_{\text{id}_Y, f} \swarrow & & \searrow (o_Y)(Ff) \\ F(\text{id}_Y f) = Ff & = & (\text{id}_{FY})(Ff) \end{array}, \quad \begin{array}{ccc} & (Ff)(F \text{id}_X) & \\ a_{f, \text{id}_X} \swarrow & & \searrow (Ff)(o_X) \\ F(f \text{id}_X) = Ff & = & (Ff) \text{id}_{FX} \end{array}$$

and

$$\begin{array}{ccc} & (Ff)F(gh) & \\ a_{f, gh} \swarrow & & \searrow (Ff)a_{g, h} \\ F(fgh) & & (Ff)(Fg)(Fh) \\ a_{fg, h} \swarrow & & \searrow a_{f, g}(Fh) \\ & F(fg)(Fh) & \end{array}$$

must commute for any 1-arrows  $f : X \rightarrow Y$ ,  $g : W \rightarrow X$ ,  $h : V \rightarrow W$  in  $\mathcal{T}$ .

Let us also explicitate what naturality of  $a_{f,g}$  means: it is equivalent to the commutativity of the diagrams

$$\begin{array}{ccc} (Ff)(Fg) \xrightarrow{a_{f,g}} F(fg) & & (Ff)(Fg) \xrightarrow{a_{f,g}} F(fg) \\ (F\varphi)(Fg) \Downarrow & & \Downarrow F(\varphi g) \text{ and } (Ff)(F\psi) \Downarrow \\ (Ff')(Fg) \xrightarrow{a_{f',g}} F(f'g) & & (Ff)(Fg') \xrightarrow{a_{f,g'}} F(fg') \end{array}$$

for any 1-arrows  $f, f' : X \rightarrow Y$ ,  $g, g' : W \rightarrow X$  and any 2-arrows  $\varphi : f \rightarrow f'$ ,  $\psi : g \rightarrow g'$  in  $\mathcal{T}$ .

A lax functor for which the 2-arrows  $o_X$  and  $a_{f,g}$  are all isomorphisms is called a *pseudofunctor*; thus for track categories these two notions are equivalent. Furthermore a pseudofunctor is called *strict* if the  $o_X$  and  $a_{f,g}$  are in fact identities. So a strict pseudofunctor is the same as a track functor, i. e. a functor enriched in the category of categories.

It is immediate from the definitions that a lax functor  $F : \mathcal{T} \rightarrow \mathcal{T}'$  induces a functor between homotopy categories  $F_{\simeq} : \mathcal{T}_{\simeq} \rightarrow \mathcal{T}'_{\simeq}$ . Moreover it clearly induces monoid

homomorphisms  $F_f : \text{End}(f) \rightarrow \text{End}(Ff)$  for each 1-arrow  $f$  in  $\mathcal{T}$ . A lax functor  $F$  between track categories is called a *lax equivalence* if the functor  $F_\simeq$  is an equivalence of categories, and all the group homomorphisms  $F_f$  are isomorphisms. It is easy to see that any lax equivalence is *locally fully faithful*, i. e. the functors  $F_{X,Y} : \llbracket X, Y \rrbracket \rightarrow \llbracket FX, FY \rrbracket$  are equivalences of groupoids.

**Proposition 2.4.1.** *Let  $D$  be a natural system on a small category  $\mathbf{C}$ , and let  $(\mathcal{T}, \tau)$ ,  $(\mathcal{T}', \tau')$  be two linear track extensions of  $\mathbf{C}$  by  $D$ . Suppose there exists a lax equivalence  $F : \mathcal{T} \rightarrow \mathcal{T}'$  which is compatible with the track extension structure in the sense that the triangles*

$$\begin{array}{ccc} & D\mathbf{f} & \\ \tau_f \swarrow & & \searrow \tau'_{Ff} \\ \text{Aut}_{\mathcal{T}}(f) & \xrightarrow{F_f} & \text{Aut}_{\mathcal{T}'}(Ff) \end{array}$$

*commute for all  $\mathbf{f} : X \rightarrow Y$  in  $\mathbf{C}$  and all 1-arrows  $f$  in  $\mathcal{T}$  with  $[f] = \mathbf{f}$ . Then  $\text{Ch}(\mathcal{T}) = \text{Ch}(\mathcal{T}') \in H^3(\mathbf{C}; D)$ .*

*Proof.* Let us recall how one constructs the characteristic class  $\text{Ch}(\mathcal{T})$ . For that, one chooses an 1-arrow  $s_{\mathbf{f}} \in \mathbf{f}$  in each homotopy class of 1-arrows in  $\mathcal{T}_\simeq = \mathbf{C}$ , and a track  $s_{\mathbf{f},g} : s_{\mathbf{f}}s_{\mathbf{g}} \Rightarrow s_{\mathbf{f}g}$  for each composable pair of morphisms in  $\mathbf{C}$ . Then a cocycle  $t$  representing the class  $\text{Ch}(\mathcal{T})$  is defined by assigning to a composable triple  $(\mathbf{f}, \mathbf{g}, \mathbf{h})$  in  $\mathbf{C}$  the element of  $D_{\mathbf{f}g\mathbf{h}}$  given by the formula

$$t(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \tau_{s_{\mathbf{f}g\mathbf{h}}}^{-1} (s_{\mathbf{f},g\mathbf{h}} + s_{\mathbf{f}}s_{\mathbf{g},\mathbf{h}} - s_{\mathbf{f},g}s_{\mathbf{h}} - s_{\mathbf{f}g,\mathbf{h}}),$$

where  $\tau_{s_{\mathbf{f}g\mathbf{h}}} : D_{\mathbf{f}g\mathbf{h}} \rightarrow \text{Aut}(s_{\mathbf{f}g\mathbf{h}})$  is the isomorphism given by the linear track extension structure of  $\mathcal{T}$ . Diagrammatically,  $t(\mathbf{f}, \mathbf{g}, \mathbf{h})$  is the element of  $D_{\mathbf{f}g\mathbf{h}}$  which corresponds under  $\tau_{s_{\mathbf{f}g\mathbf{h}}}$  to the automorphism of  $s_{\mathbf{f}g\mathbf{h}}$  given by the counterclockwise roundtrip in the diagram

$$\begin{array}{ccc} & s_{\mathbf{f}}s_{\mathbf{g}\mathbf{h}} & \\ s_{\mathbf{f},g\mathbf{h}} \swarrow & & \swarrow s_{\mathbf{f}}s_{\mathbf{g},\mathbf{h}} \\ s_{\mathbf{f}g\mathbf{h}} & & s_{\mathbf{f}}s_{\mathbf{g}}s_{\mathbf{h}} \\ s_{\mathbf{f}g,\mathbf{h}} \swarrow & & \swarrow s_{\mathbf{f},g}s_{\mathbf{h}} \\ & s_{\mathbf{f}g\mathbf{h}} & \end{array}$$

Given now a lax equivalence  $(F, o, a)$  from  $\mathcal{T}$  to  $\mathcal{T}'$  and a choice of  $s_{\mathbf{f}}$ ,  $s_{\mathbf{f},g}$  for  $\mathcal{T}$  as above, we can make similar choices for  $\mathcal{T}'$  by defining  $s'_{\mathbf{f}} = Fs_{\mathbf{f}}$  and determining  $s'_{\mathbf{f},g}$  by the commutative diagrams

$$\begin{array}{ccc} & F(s_{\mathbf{f}}s_{\mathbf{g}}) & \\ F(s_{\mathbf{f},g}) \swarrow & & \swarrow a_{s_{\mathbf{f}},s_{\mathbf{g}}} \\ Fs_{\mathbf{f}g} & \xleftarrow{s'_{\mathbf{f},g}} & (Fs_{\mathbf{f}})(Fs_{\mathbf{g}}) \end{array}$$

Thus the value of a cocycle  $t'$  for  $\text{Ch}(\mathcal{T}')$  on a triple  $\mathbf{f}, \mathbf{g}, \mathbf{h}$  is determined by the outer roundtrip in the diagram

$$\begin{array}{ccccc}
 & & F s_{\mathbf{f}} F s_{\mathbf{g}} \mathbf{h} & & \\
 & \swarrow a_{s_{\mathbf{f}}, s_{\mathbf{g}} \mathbf{h}} & & \nwarrow F s_{\mathbf{f}} F s_{\mathbf{g}, \mathbf{h}} & \\
 & F(s_{\mathbf{f}} s_{\mathbf{g}} \mathbf{h}) & & F s_{\mathbf{f}} F(s_{\mathbf{g}} s_{\mathbf{h}}) & \\
 & \swarrow F s_{\mathbf{f}, \mathbf{g}} \mathbf{h} & & \nwarrow F s_{\mathbf{f}} a_{s_{\mathbf{g}}, s_{\mathbf{h}}} & \\
 F s_{\mathbf{f}} \mathbf{g} \mathbf{h} & & F(s_{\mathbf{f}} s_{\mathbf{g}} s_{\mathbf{h}}) & & F s_{\mathbf{f}} F s_{\mathbf{g}} F s_{\mathbf{h}} \\
 & \swarrow F s_{\mathbf{f}, \mathbf{g}, \mathbf{h}} & & \nwarrow a_{s_{\mathbf{f}}, s_{\mathbf{g}} s_{\mathbf{h}}} & \\
 & F(s_{\mathbf{f}, \mathbf{g}} s_{\mathbf{h}}) & & F(s_{\mathbf{f}} s_{\mathbf{g}} s_{\mathbf{h}}) & \\
 & \swarrow F s_{\mathbf{f}, \mathbf{g}, \mathbf{h}} & & \nwarrow a_{s_{\mathbf{f}}, s_{\mathbf{g}}, s_{\mathbf{h}}} & \\
 & F(s_{\mathbf{f}} \mathbf{g} s_{\mathbf{h}}) & & F(s_{\mathbf{f}} s_{\mathbf{g}}) F s_{\mathbf{h}} & \\
 & \swarrow a_{s_{\mathbf{f}}, s_{\mathbf{g}}, s_{\mathbf{h}}} & & \nwarrow F s_{\mathbf{f}, \mathbf{g}} F s_{\mathbf{h}} & \\
 & F s_{\mathbf{f}} \mathbf{g} F s_{\mathbf{h}} & & & 
 \end{array}$$

In this diagram, upper and lower squares commute by naturality of  $a$ , and the right square commutes as an instance of the coherence condition. It thus follows that  $t'(\mathbf{f}, \mathbf{g}, \mathbf{h})$  is given by the roundtrip of the left square, i. e.

$$t'(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \tau'_{F s_{\mathbf{f}} \mathbf{g} \mathbf{h}}^{-1} (F s_{\mathbf{f}, \mathbf{g}} \mathbf{h} + F(s_{\mathbf{f}} s_{\mathbf{g}} \mathbf{h}) - F(s_{\mathbf{f}, \mathbf{g}} s_{\mathbf{h}}) - F s_{\mathbf{f}} \mathbf{g} \mathbf{h}).$$

Now recall that each  $F_{X,Y} : \llbracket X, Y \rrbracket \rightarrow \llbracket F X, F Y \rrbracket$  is a functor, hence we can write

$$t'(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \tau'_{F s_{\mathbf{f}} \mathbf{g} \mathbf{h}}^{-1} (F(s_{\mathbf{f}, \mathbf{g}} \mathbf{h} + s_{\mathbf{f}} s_{\mathbf{g}, \mathbf{h}} - s_{\mathbf{f}, \mathbf{g}} s_{\mathbf{h}} - s_{\mathbf{f}} \mathbf{g} \mathbf{h})).$$

It then follows from compatibility of  $F$  with the linear track extension structures that we obtained a cocycle  $t'$  that actually coincides with  $t$ .  $\square$

Our next aim is to prove the converse of the above proposition, namely, that if two linear track extensions have the same characteristic class, then there is a lax equivalence between them.

For this, let us define for a track category  $\mathcal{T} = (\mathcal{T}_1 \rightrightarrows \mathcal{T}_0)$  its *relaxation*, which is a track category  $\tilde{\mathcal{T}} = (\tilde{\mathcal{T}}_1 \rightrightarrows \tilde{\mathcal{T}}_0)$  equipped with a weak equivalence  $E^{\mathcal{T}} : \tilde{\mathcal{T}} \rightarrow \mathcal{T}$  (NB: by definition, *weak* equivalences are *strict* functors, as opposed to more general *lax* equivalences). We put  $\tilde{\mathcal{T}}_0 = \mathbf{P}(\mathcal{T}_0)$ , the path category of  $\mathcal{T}_0$ . Recall that for a graph  $G$  its path category  $\mathbf{P}G$  is the free category on  $G$ . Thus objects of  $\mathbf{P}G$  are nodes of  $G$ , and morphisms of  $\mathbf{P}G$  are finite composable sequences

$$X_0 \xleftarrow{x_1} X_1 \xleftarrow{\cdots} \xleftarrow{x_n} X_n,$$

$n \geq 0$ , of arrows of  $G$ , identities being empty sequences and composition given by concatenation. Thus for a small category  $\mathbf{C}$  considered as a graph there is a canonical functor  $E^{\mathbf{C}} : \mathbf{P}\mathbf{C} \rightarrow \mathbf{C}$  which is identity on objects, given by sending a sequence to its composite in  $\mathbf{C}$  (and a sequence with  $n = 0$  to the identity of the corresponding object). In particular  $\tilde{\mathcal{T}}_0$  comes equipped with such a canonical functor  $E^{\mathcal{T}_0} : \tilde{\mathcal{T}}_0 \rightarrow \mathcal{T}_0$ . We then define the track

structure of  $\tilde{\mathcal{T}}$  by pulling it back from  $\mathcal{T}$  along  $E^{\mathcal{T}_0}$ ; that is, we define for two sequences of the form

$$\begin{array}{c}
 \dots \quad \swarrow \quad \nwarrow \quad \dots \\
 X_1 \quad \swarrow \quad \nwarrow \quad X_{n-1} \\
 x_1 \quad \swarrow \quad \nwarrow \quad x_n \\
 X_0 \quad \swarrow \quad \nwarrow \quad X_n \\
 \parallel \quad \swarrow \quad \nwarrow \quad \parallel \\
 B \quad \swarrow \quad \nwarrow \quad A \\
 \parallel \quad \swarrow \quad \nwarrow \quad \parallel \\
 Y_0 \quad \swarrow \quad \nwarrow \quad Y_m \\
 y_1 \quad \swarrow \quad \nwarrow \quad y_m \\
 Y_1 \quad \swarrow \quad \nwarrow \quad Y_{m-1} \\
 \dots \quad \swarrow \quad \nwarrow \quad \dots
 \end{array}$$

the set of tracks between them by the formula

$$\mathrm{Hom}_{\llbracket A, B \rrbracket_{\tilde{\mathcal{T}}}}((x_1, \dots, x_n), (y_1, \dots, y_m)) = \mathrm{Hom}_{\llbracket A, B \rrbracket_{\mathcal{T}}}(x_1 \dots x_n, y_1 \dots y_m).$$

Then trivially one checks that this determines a track category, that  $E^{\mathcal{T}_0}$  extends to a strict functor  $E^{\mathcal{T}} : \tilde{\mathcal{T}} \rightarrow \mathcal{T}$ , that it induces an equivalence (in fact, an isomorphism)  $\tilde{\mathcal{T}} \rightarrow \tilde{\mathcal{T}}_{\sim}$ , and the induced homomorphisms  $\mathrm{Aut}_{\tilde{\mathcal{T}}}((f_1, \dots, f_n)) \rightarrow \mathrm{Aut}_{\mathcal{T}}(f_1 \dots f_n)$  are all isomorphisms. In other words,  $E^{\mathcal{T}}$  is a weak equivalence.

Observe also that there is moreover a canonical lax equivalence  $(F^{\mathcal{T}}, o^{\mathcal{T}}, a^{\mathcal{T}}) : \mathcal{T} \rightarrow \tilde{\mathcal{T}}$ , given as follows:

- for an 1-arrow  $f : X \rightarrow Y$  in  $\mathcal{T}$ ,  $F_{X,Y}^{\mathcal{T}}(f)$  is the 1-tuple  $(f) : X \rightarrow Y$  in  $\tilde{\mathcal{T}}$ ;
- for a 2-arrow  $\varphi : f \Rightarrow g$ , with  $f, g : X \rightarrow Y$  in  $\mathcal{T}$ ,  $F_{X,Y}^{\mathcal{T}}(\varphi)$  is the same 2-arrow  $\varphi \in \mathrm{Hom}_{\llbracket X, Y \rrbracket_{\mathcal{T}}}(f, g) = \mathrm{Hom}_{\llbracket X, Y \rrbracket_{\tilde{\mathcal{T}}}}((f), (g))$ ;
- for  $X \in \mathrm{Ob} \mathbf{C}$ ,  $o_X^{\mathcal{T}} : () \Rightarrow (\mathrm{id}_X)$  is given by  $\mathrm{id}_{\mathrm{id}_X} \in \mathrm{Hom}_{\llbracket X, X \rrbracket_{\mathcal{T}}}(\mathrm{id}_X, \mathrm{id}_X) = \mathrm{Hom}_{\llbracket X, X \rrbracket_{\tilde{\mathcal{T}}}}((\mathrm{id}_X), (\mathrm{id}_X))$ ;
- for 1-arrows  $f : X \rightarrow Y$ ,  $g : W \rightarrow X$  in  $\mathbf{C}$ ,  $a_{f,g}^{\mathcal{T}} : (f, g) \Rightarrow (fg)$  is  $\mathrm{id}_{fg} \in \mathrm{Hom}_{\llbracket X, Y \rrbracket_{\mathcal{T}}}(fg, fg) = \mathrm{Hom}_{\llbracket X, Y \rrbracket_{\tilde{\mathcal{T}}}}((f, g), (fg))$ .

It is straightforward to check that this indeed defines a lax equivalence. One notes that the composite of a strict functor and a pseudofunctor is well defined and in fact  $E^{\mathcal{T}} F^{\mathcal{T}}$  is identity.

We now have

**Theorem 2.4.2.** *Let  $\mathcal{T}, \mathcal{T}'$  be linear track extensions of a small category  $\mathbf{C}$  by a natural system  $D$ . If  $\mathrm{Ch}(\mathcal{T}) = \mathrm{Ch}(\mathcal{T}') \in H^3(\mathbf{C}; D)$ , then there exists a lax equivalence  $\mathcal{T} \rightarrow \mathcal{T}'$ .*

*Proof.* Let us begin by assigning to an 1-arrow  $f : X \rightarrow Y$  in  $\mathcal{T}$  an 1-arrow  $S(f) : X \rightarrow Y$  in  $\mathcal{T}'$  in such a way that  $[S(f)] = [f] : X \rightarrow Y$  in  $\mathbf{C}$ . Since  $\tilde{\mathcal{T}}_0 = \mathbf{P}\mathcal{T}_0$  is a free category, this assignment extends uniquely to a functor  $S : \tilde{\mathcal{T}}_0 \rightarrow \mathcal{T}'_0$ . Let us denote by  $S^* \mathcal{T}'$  the track category with  $(S^* \mathcal{T}')_0 = \tilde{\mathcal{T}}_0$  obtained by pulling back the 2-arrows from  $\mathcal{T}'$  along  $S$ , just as we did when defining  $\tilde{\mathcal{T}}$ . More precisely, define

$$\mathrm{Hom}_{\llbracket X, Y \rrbracket_{S^* \mathcal{T}'}}((x_1, \dots, x_n), (y_1, \dots, y_m)) = \mathrm{Hom}_{\llbracket X, Y \rrbracket_{\mathcal{T}'}}(S(x_1) \dots S(x_n), S(y_1) \dots S(y_m)).$$



Then, exactly as before, one sees that  $S$  extends to a strict functor  $S : S^* \mathcal{T}' \rightarrow \mathcal{T}'$  which is a weak equivalence.

We have now elements  $\text{ch}(\mathcal{T}) \in H^3(\mathbf{C}, \mathcal{T}_0; D)$ ,  $\text{ch}(\mathcal{T}') \in H^3(\mathbf{C}, \mathcal{T}'_0; D)$  and  $\text{ch}(\tilde{\mathcal{T}})$ ,  $\text{ch}(S^* \mathcal{T}') \in H^3(\mathbf{C}, \mathbf{P} \mathcal{T}_0; D)$ , such that in the diagram of cohomology groups

$$\begin{array}{ccccc} H^3(\mathbf{C}, \mathcal{T}_0; D) & \xrightarrow{E^*} & H^3(\mathbf{C}, \mathbf{P} \mathcal{T}_0; D) & \xleftarrow{S^*} & H^3(\mathbf{C}, \mathcal{T}'_0; D) \\ & \searrow \partial & \downarrow \partial^\cong & \swarrow \partial' & \\ & & H^3(\mathbf{C}; D) & & \end{array}$$

one has

$$\begin{aligned} \partial^\cong \text{ch}(\tilde{\mathcal{T}}) &= \partial^\cong E^* \text{ch}(\mathcal{T}) = \partial \text{ch}(\mathcal{T}) = \text{Ch}(\mathcal{T}) \\ &= \text{Ch}(\mathcal{T}') = \partial' \text{ch}(\mathcal{T}') = \partial^\cong S^* \text{ch}(\mathcal{T}') = \partial^\cong \text{ch}(S^* \mathcal{T}'). \end{aligned}$$

Since cohomology of a free category vanishes in dimensions  $\geq 2$  [9], it follows from the long exact sequence connecting the relative and absolute cohomology groups, that  $\partial^\cong$  is an isomorphism. Thus  $\text{ch}(\tilde{\mathcal{T}}) = \text{ch}(S^* \mathcal{T}') \in H^3(\mathbf{C}, \mathbf{P} \mathcal{T}_0; D)$ . Then there exists an isomorphism of relative extensions  $s : \tilde{\mathcal{T}} \rightarrow S^* \mathcal{T}'$ , and precomposing it with  $S$  we obtain a weak equivalence  $Ss : \tilde{\mathcal{T}} \rightarrow \mathcal{T}'$ . It then remains to compose this with the lax equivalence  $F^{\mathcal{T}} : \mathcal{T} \rightarrow \tilde{\mathcal{T}}$  to obtain the required lax equivalence  $SsF^{\mathcal{T}} : \mathcal{T} \rightarrow \mathcal{T}'$ .  $\square$

**2.5. Linear extensions and second cohomology of categories.** To have a more complete picture of the rôle of cohomology of small categories, let us recall the definition of *linear extensions of categories* and their relationship with the second cohomology following [9]. Let  $D$  be a natural system on a small category  $\mathbf{C}$ . A linear extension

$$0 \rightarrow D \rightarrow \mathbf{E} \xrightarrow{p} \mathbf{C} \rightarrow 0$$

of  $\mathbf{C}$  by  $D$  is a category  $\mathbf{E}$ , a full functor  $p$  which is identity on objects, and, moreover, for each morphism  $f : A \rightarrow B$  in  $\mathbf{C}$ , a transitive and effective action of the abelian group  $D_f$  on the subset  $p^{-1}(f) \subseteq \text{Hom}_{\mathbf{E}}(A, B)$ ,

$$D_f \times p^{-1}(f) \rightarrow p^{-1}(f); \quad (a, \tilde{f}) \mapsto a + \tilde{f},$$

such that the following identity holds

$$(a + \tilde{f})(b + \tilde{g}) = fb + ag + \tilde{f}\tilde{g}.$$

Here  $f$  and  $g$  are two composable arrows in  $\mathbf{C}$ ,  $\tilde{f} \in p^{-1}(f)$ ,  $\tilde{g} \in p^{-1}(g)$  and  $a \in D_f$ ,  $b \in D_g$ . Two linear extensions  $\mathbf{E}$  and  $\mathbf{E}'$  are *equivalent* if there is an isomorphism of categories  $\epsilon : \mathbf{E} \rightarrow \mathbf{E}'$  with  $p'\epsilon = p$  and with  $\epsilon(a + \tilde{f}) = a + \epsilon(\tilde{f})$ .

Let  $\text{Linext}(\mathbf{C}; D)$  be the set of equivalence classes of linear extensions of  $\mathbf{C}$  by  $D$ .

**Theorem 2.5.1.** ([9]) *There is a natural bijection*

$$\text{Linext}(\mathbf{C}; D) \approx H^2(\mathbf{C}; D).$$

### 3. LOCAL AND GLOBAL Ext-GROUPS

**3.1. The local-global spectral sequence.** Let  $\mathbf{I}$  be a small category and let  $R : \mathbf{I} \rightarrow \mathcal{Rings}$  be a functor to the category of rings with unit. A *left  $R$ -module* is a functor  $M : \mathbf{I} \rightarrow \mathcal{A}$  together with left  $R_i$ -module structures on abelian groups  $M_i$ ,  $i \in \text{Ob}(\mathbf{I})$ , such that for any arrow  $\chi : i \rightarrow j$  and any  $r \in R_i$ ,  $m \in M_i$  one has

$$R_\chi(rm) = R_\chi(r)M_\chi(m)$$

in  $M_j$ . We denote the category of all left modules over a ring-valued functor  $R : \mathbf{I} \rightarrow \mathcal{Rings}$  by  $R\text{-mod}$ .

As an example, we can take any small subcategory  $\mathbf{I}$  of the category of commutative rings and let  $\mathcal{O}$  be the inclusion  $\mathbf{I} \hookrightarrow \mathcal{Rings}$ . Thus  $\mathcal{O}$  is a ring valued functor. For any ring  $S \in \mathbf{I}$  the absolute Kähler differentials  $\Omega_S^*$  is a module over  $S$ . Since  $\Omega_S^*$  functorially depends on  $S$  we obtain that  $\Omega^* \in \mathcal{O}\text{-mod}$ . Another example comes from topology. Let  $\mathbf{I}$  be a small subcategory of the category of topological spaces. Then for any ring  $R$ , the ordinary (singular) cohomology of spaces with coefficients in  $R$  defines a ring valued functor  $H^*(\_, R)$ , and for any  $R$ -module  $M$  the functor  $H^*(\_, M)$  is a module over  $H^*(\_, R)$  in the above sense. Similarly  $X \mapsto \mathbb{Z}[\pi_1 X]$  is a ring valued functor defined on any small subcategory of the category of pointed topological spaces, while  $X \mapsto \pi_i X$  is a module over it, for any  $i \geq 2$ .

It is well known that the category  $R\text{-mod}$  is an abelian category. Moreover it has enough projective and injective objects (see also Section 3.3). For any  $R$ -modules  $M$  and  $N$  one defines the natural systems  $\mathcal{H}om_R(M, N)$  and  $\mathcal{E}xt_R^n(M, N)$  on  $\mathbf{I}$  by

$$\mathcal{H}om_R(M, N)_{i \xrightarrow{\chi} j} = \text{Hom}_{R_i}(M_i, N_j)$$

and

$$\mathcal{E}xt_R^n(M, N)_{i \xrightarrow{\chi} j} = \text{Ext}_{R_i}^n(M_i, N_j)$$

respectively, where the actions of  $R_i$  on  $N_j$  are given via restriction of scalars along  $R_\chi : R_i \rightarrow R_j$ . We call the natural systems  $\mathcal{H}om_R(M, N)$  and  $\mathcal{E}xt_R^n(M, N)$  *local Hom and local Ext groups*. One observes that in the case when  $R$  is a constant functor, these natural systems actually come from bifunctors. The following theorem, which is the main result of this section, was proved for the particular case of such constant  $R$  in [13].

**Theorem 3.1.1. (the local-to-global spectral sequence)** *Let  $\mathbf{I}$  be a small category and let  $R : \mathbf{I} \rightarrow \mathcal{Rings}$  be a functor to the category of rings with unit. For any  $R$ -modules  $M$  and  $N$  there exists a spectral sequence with*

$$E_2^{pq} = H^p(\mathbf{I}; \mathcal{E}xt_R^q(M, N)) \implies \text{Ext}_{R\text{-mod}}^{p+q}(M, N).$$

*The result remains true also with rings replaced by ringoids.*

The last statement about ringoid-valued functors is essential to prove our main theorem on strengthening of track theories. We refer the reader to Section 3.2 for the definition of

ringoids and related stuff, and to page 24 for the proof. Before we go into more detail let us give some useful consequences.

**Corollary 3.1.2.** *Let  $\mathbf{I}$  be a small category and let  $M, N$  be  $\mathcal{R}$ -modules, where*

$$\mathcal{R} : \mathbf{I} \rightarrow \mathcal{Ringoids}$$

*is a functor. Then one has a five-term exact sequence*

$$\begin{aligned} 0 \rightarrow H^1(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) &\rightarrow \text{Ext}_{\mathcal{R}\text{-}\mathbf{mod}}^1(M, N) \\ &\rightarrow H^0(\mathbf{I}; \mathcal{E}xt_{\mathcal{R}}^1(M, N)) \rightarrow H^2(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) \rightarrow \text{Ext}_{\mathcal{R}\text{-}\mathbf{mod}}^2(M, N). \end{aligned}$$

*Moreover, if  $\text{gl. dim } \mathcal{R}_i \leq 1$  for each object  $i$ , then one has an exact sequence*

$$\begin{aligned} 0 \rightarrow H^1(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) &\rightarrow \cdots \rightarrow H^n(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) \\ &\rightarrow \text{Ext}_{\mathcal{R}\text{-}\mathbf{mod}}^n(M, N) \rightarrow H^{n-1}(\mathbf{I}; \mathcal{E}xt_{\mathcal{R}}^1(M, N)) \rightarrow H^{n+1}(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) \rightarrow \cdots \end{aligned}$$

**Corollary 3.1.3.** *Suppose  $M_i$  is a projective  $\mathcal{R}_i$ -module for each  $i \in \text{Ob}(\mathbf{I})$ . Then there is an isomorphism*

$$H^*(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) \cong \text{Ext}_{\mathcal{R}\text{-}\mathbf{mod}}^*(M, N).$$

**3.2. Ringoids and modules over them.** In this subsection we recall some well known facts about ringoids and modules over them. A good reference on this subject is [16].

A *ringoid* is a category enriched in abelian groups. It is thus a small category  $\mathcal{R}$  together with the structure of abelian group on its Hom-sets in such a way that composition is biadditive. Morphisms of ringoids are enriched functors, i. e. functors preserving the abelian group structures. These are also called *additive* functors. The category of ringoids will be denoted by  $\mathcal{Ringoids}$ .

Let  $\mathcal{R}$  be a ringoid. We denote by  $\mathcal{R}\text{-}\mathbf{mod}$  the category of all covariant additive functors from  $\mathcal{R}$  to  $\mathcal{M}$ , and by  $\mathbf{mod}\text{-}\mathcal{R}$  the category of all contravariant additive functors from  $\mathcal{R}$  to  $\mathcal{M}$ . Objects from  $\mathcal{R}\text{-}\mathbf{mod}$  are called *left modules* over  $\mathcal{R}$ , while those from  $\mathbf{mod}\text{-}\mathcal{R}$  are called *right modules*.

For any small category  $\mathbf{I}$ , we let  $\mathbb{Z}[\mathbf{I}]$  be the ringoid with the same objects as  $\mathbf{I}$ , while for any objects  $i$  and  $j$  the group of homomorphisms from  $i$  to  $j$  in  $\mathbb{Z}[\mathbf{I}]$  is the free abelian group generated by  $\text{Hom}_{\mathbf{I}}(i, j)$ :

$$\text{Hom}_{\mathbb{Z}[\mathbf{I}]}(i, j) = \mathbb{Z}[\text{Hom}_{\mathbf{I}}(i, j)],$$

whereas the composition law is induced by

$$\mathbb{Z}[\text{Hom}_{\mathbf{I}}(i, j)] \otimes \mathbb{Z}[\text{Hom}_{\mathbf{I}}(j, k)] \cong \mathbb{Z}[\text{Hom}_{\mathbf{I}}(i, j) \times \text{Hom}_{\mathbf{I}}(j, k)] \rightarrow \mathbb{Z}[\text{Hom}_{\mathbf{I}}(i, k)].$$

Then clearly one has  $\mathbb{Z}[\mathbf{I}]\text{-}\mathbf{mod} \simeq \mathcal{M}^{\mathbf{I}}$ .

For any ringoid  $\mathcal{R}$  and an object  $c \in \mathcal{R}$  we define  $h_c : \mathcal{R} \rightarrow \mathcal{M}$  and  $h^c : \mathcal{R}^{\text{op}} \rightarrow \mathcal{M}$  by

$$h_c(x) = \text{Hom}_{\mathcal{R}}(c, x)$$

and

$$h^c(x) = \text{Hom}_{\mathcal{R}}(x, c).$$

Then one has natural isomorphisms

$$\text{Hom}_{\mathcal{R}\text{-mod}}(h_c, M) \cong M(c)$$

and

$$\text{Hom}_{\text{mod-}\mathcal{R}}(h^c, N) \cong N(c).$$

Therefore, the family of objects  $(h_c)_{c \in \text{Ob}(\mathcal{R})}$  (resp.  $(h^c)_{c \in \text{Ob}(\mathcal{R})}$ ) forms a family of small projective generators in  $\mathcal{R}\text{-mod}$  (resp. in  $\text{mod-}\mathcal{R}$ ). The functor  $h_c$  is called *the standard free left  $\mathcal{R}$ -module concentrated at  $c$* .

Let  $M : \mathcal{R} \rightarrow \mathcal{M}$  and  $N : \mathcal{R}^{\text{op}} \rightarrow \mathcal{M}$  be additive functors. Let  $N \otimes_{\mathcal{R}} M$  be the abelian group defined by

$$\left( \bigoplus_{c \in \text{Ob}(\mathcal{R})} N(c) \otimes M(c) \right) / \sim.$$

Here  $\sim$  is the congruence generated by

$$N(\alpha)x \otimes y \sim x \otimes M(\alpha)y$$

where  $\alpha : c_1 \rightarrow c$  is a morphism in  $\mathcal{R}$ ,  $x \in N(c)$ , and  $y \in M(c_1)$ .

Then one has isomorphisms

$$h^c \otimes_{\mathcal{R}} M \cong M(c),$$

$$N \otimes_{\mathcal{R}} h_c \cong N(c).$$

Let  $f : \mathcal{R} \rightarrow \mathcal{S}$  be a morphism of ringoids. Composition with  $f$  induces a functor

$$f^* : \mathcal{S}\text{-mod} \rightarrow \mathcal{R}\text{-mod}.$$

It is well known that  $f^*$  has right and left adjoint functors  $f_*$  and  $f_!$  respectively (the so-called right and left Kan extensions) and for any  $F : \mathcal{R} \rightarrow \mathcal{M}$  one has isomorphisms

$$(f_* F)(d) \cong \text{Hom}_{\mathcal{R}\text{-mod}}(f^* h_d, F),$$

$$(f_! F)(d) \cong f^* h^d \otimes_{\mathcal{R}} F.$$

**3.3. Modules over a ringoid valued functor.** Let us consider now a small category  $\mathbf{I}$  and a covariant functor

$$\mathcal{R} : \mathbf{I} \rightarrow \text{Ringoids}.$$

We introduce a category  $\int_{\mathbf{I}} \mathcal{R}$  or simply  $\int \mathcal{R}$  as follows. Objects of  $\int \mathcal{R}$  are pairs  $(i, x)$ , where  $i$  is an object of  $\mathbf{I}$  and  $x$  is an object of  $\mathcal{R}_i$ . A morphism  $(i, x) \rightarrow (j, y)$  is a pair  $(\alpha, r)$ , where  $\alpha : i \rightarrow j$  is a morphism in  $\mathbf{I}$  and  $r : \mathcal{R}_{\alpha}(x) \rightarrow y$  is a morphism in  $\mathcal{R}_j$ . Composition in  $\int \mathcal{R}$  is defined by

$$(\alpha, r) \circ (\beta, s) = (\alpha \circ \beta, r \circ \mathcal{R}_{\alpha}(s)).$$

Then for each  $i \in \mathbf{I}$  we have an obvious functor  $\xi_i : \mathcal{R}_i \rightarrow \int \mathcal{R}$  which assigns  $(i, x)$  to an object  $x \in \text{Ob}(\mathcal{R}_i)$ .

We will say that  $M$  is a *left  $\mathcal{R}$ -module* if the following data are given:

- i) a left  $\mathcal{R}_i$ -module  $M_i$  for each object  $i \in \mathbf{I}$ ;
- ii) a homomorphism  $M_\alpha : M_i \rightarrow \mathcal{R}_\alpha^* M_j$  of  $\mathcal{R}_i$ -modules for each arrow  $\alpha : i \rightarrow j$  of  $\mathbf{I}$ .

Moreover it is required that for any composable morphisms  $\alpha$  and  $\beta$  one has  $M_{\alpha\beta} = M_\alpha M_\beta$ .

If  $M$  is a left  $\mathcal{R}$ -module,  $i$  is an object of  $\mathbf{I}$ , and  $x$  is an object of the ringoid  $\mathcal{R}_i$ , then we denote by  $M_{(i,x)}$  the value  $M_i(x)$  of  $M_i$  on  $x$ . Having this in mind it is clear that a left  $\mathcal{R}$ -module is nothing else but a functor  $M : \int \mathcal{R} \rightarrow \mathcal{M}$  such that each composition  $M \circ \xi_i : \mathcal{R}_i \rightarrow \mathcal{M}$ ,  $i \in \mathbf{I}$ , is an additive functor. The category of all left  $\mathcal{R}$ -modules will be denoted by  $\mathcal{R}\text{-mod}$ .

Yet another description of this category is possible, showing that  $\mathcal{R}\text{-mod}$  is itself equivalent to the category of modules over a single ringoid. Given a functor  $\mathcal{R} : \mathbf{I} \rightarrow \text{Ringoids}$  as above, we define its *total ringoid*  $\mathcal{R}[\mathbf{I}]$  in the following way: the set  $\text{Ob}(\mathcal{R}[\mathbf{I}])$  of objects of the ringoid  $\mathcal{R}[\mathbf{I}]$  is the disjoint union  $\coprod_{i \in \text{Ob}(\mathbf{I})} \text{Ob}(\mathcal{R}_i)$  — or else again the set of pairs  $(i, x)$ , just as for  $\int \mathcal{R}$ . Morphisms of the ringoid  $\mathcal{R}[\mathbf{I}]$  are given by

$$\text{Hom}_{\mathcal{R}[\mathbf{I}]}((i, x), (j, y)) = \bigoplus_{i \xrightarrow{\alpha} j} \text{Hom}_{\mathcal{R}_j}(\mathcal{R}_\alpha(x), y).$$

Composition homomorphisms are given by

$$\begin{aligned} & \left( \bigoplus_{i \xrightarrow{\alpha} j} \text{Hom}_{\mathcal{R}_j}(\mathcal{R}_\alpha(x), y) \right) \otimes \left( \bigoplus_{j \xrightarrow{\beta} k} \text{Hom}_{\mathcal{R}_k}(\mathcal{R}_\beta(y), z) \right) \\ & \xrightarrow{\cong} \bigoplus_{i \xrightarrow{\alpha} j \xrightarrow{\beta} k} \text{Hom}_{\mathcal{R}_j}(\mathcal{R}_\alpha(x), y) \otimes \text{Hom}_{\mathcal{R}_k}(\mathcal{R}_\beta(y), z) \\ & \xrightarrow{\oplus_{\alpha, \beta} \mathcal{R}_\beta \otimes 1} \bigoplus_{i \xrightarrow{\alpha} j \xrightarrow{\beta} k} \text{Hom}_{\mathcal{R}_k}(\mathcal{R}_\beta \mathcal{R}_\alpha(x), \mathcal{R}_\beta y) \otimes \text{Hom}_{\mathcal{R}_k}(\mathcal{R}_\beta(y), z) \\ & \xrightarrow{\oplus_{\alpha, \beta} \circ} \bigoplus_{i \xrightarrow{\alpha} j \xrightarrow{\beta} k} \text{Hom}_{\mathcal{R}_k}(\mathcal{R}_\beta \mathcal{R}_\alpha(x), z) \rightarrow \bigoplus_{i \xrightarrow{\gamma} k} \text{Hom}_{\mathcal{R}_k}(\mathcal{R}_\gamma(x), z), \end{aligned}$$

and the identity of  $x \in \text{Ob}(\mathcal{R}_i)$  is the element of  $\bigoplus_{i \xrightarrow{\varepsilon} i} \text{Hom}_{\mathcal{R}_i}(\mathcal{R}_\varepsilon(x), x)$  given by the identity of  $x$  in  $\mathcal{R}_i$ , situated in the  $\text{id}_i$ -th summand. It is straightforward to check that this construction indeed yields a ringoid. One then has

**Proposition 3.3.1.** *For any ringoid-valued functor  $\mathcal{R} : \mathbf{I} \rightarrow \text{Ringoids}$ , the category of left  $\mathcal{R}$ -modules is equivalent to  $\mathcal{R}[\mathbf{I}]\text{-mod}$ .*

*Proof.* An  $\mathcal{R}[\mathbf{I}]$ -module  $M$  is a family of abelian groups  $(M_{(i,x)})_{x \in \coprod_i \text{Ob}(\mathcal{R}_i)}$  and a family of abelian group homomorphisms

$$\left( \bigoplus_{i \xrightarrow{\alpha} j} \text{Hom}_{\mathcal{R}_j}(\mathcal{R}_\alpha(x), y) \xrightarrow{M_{(i,x), (j,y)}} \text{Hom}_{\mathcal{M}}(M_{(i,x)}, M_{(j,y)}) \right)_{x \in \text{Ob}(\mathcal{R}_i), y \in \text{Ob}(\mathcal{R}_j)},$$

satisfying certain conditions. Just by universality of sums then, specifying the above homomorphisms  $M_{(i,x),(j,y)}$  is equivalent to specifying families

$$\left( \text{Hom}_{\mathcal{R}_j}(\mathcal{R}_\alpha(x), y) \xrightarrow{M_\alpha} \text{Hom}_{\mathcal{R}}(M_{(i,x)}, M_{(j,y)}) \right)_{\alpha \in \text{Hom}_{\mathcal{I}}((i,x),(j,y))}.$$

It is then straightforward to check that the conditions on the  $M_{(i,x),(j,y)}$  to form an  $\mathcal{R}[\mathbf{I}]$ -module give precisely the conditions on the  $M_\alpha$  to form an  $\mathcal{R}$ -module.  $\square$

*Example 3.3.2.* Since rings with unit are the same as ringoids with single object, as a particular case of the above construction we have a description of the category of modules over a ring-valued functor  $R : \mathbf{I} \rightarrow \mathcal{Rings}$  as in 3.1. Namely, one has an equivalence  $R\text{-mod} \simeq R[\mathbf{I}]\text{-mod}$ , where  $R[\mathbf{I}]$  is a ringoid having the same objects as  $\mathbf{I}$ , with

$$\text{Hom}_{R[\mathbf{I}]}(i, j) = \bigoplus_{\alpha \in \text{Hom}_{\mathbf{I}}(i, j)} R_j,$$

composition given by

$$\left( \sum_{j \xrightarrow{\beta} k} x_\beta \right) \circ \left( \sum_{i \xrightarrow{\alpha} j} y_\alpha \right) = \sum x_\beta R_\beta(y_\alpha).$$

To obtain something really familiar, take the further particular case of this, when  $\mathbf{I}$  is a group  $G$ , considered as a category with one object. Then a ring-valued functor  $R$  on this category is the same as a ring  $R$  with a  $G$ -action, and a module  $M$  over this functor is the same as a  $G$ -equivariant  $R$ -module, i. e. an  $R$ -module with a  $G$ -action such that

$$(rm)^g = r^g m^g$$

for any  $r \in R$ ,  $m \in M$ ,  $g \in G$ . Furthermore  $R[G]$  is in this case none other than the *crossed group algebra*, i. e. the ring obtained by freely adjoining to the multiplicative monoid of  $R$  the group  $G$  subject to the commutation relations  $rg = gr^g$  for all  $r \in R$ ,  $g \in G$ . That  $G$ -equivariant  $R$ -modules are the same as  $R[G]$ -modules is a classical fact.

It is thus clear that  $\mathcal{R}\text{-mod}$  is an abelian category with enough projective and injective objects. Let us give the explicit description of the projective generators and injective cogenerators corresponding to the standard ones from  $\mathcal{R}[\mathbf{I}]$ .

Take  $i \in \text{Ob}(\mathbf{I})$  and let  $x$  be an object of the ringoid  $\mathcal{R}_i$ . Then, in accord with the above 3.3.1, associated to the standard free  $\mathcal{R}[\mathbf{I}]$ -module concentrated at  $(i, x)$  there is a left  $\mathcal{R}$ -module  $h_{i,x}^{\mathcal{R}}$  given by

$$(h_{i,x}^{\mathcal{R}})_j(y) = \bigoplus_{i \xrightarrow{\alpha} j} \text{Hom}_{\mathcal{R}_j}(\mathcal{R}_\alpha(x), y).$$

In other words  $(h_{i,x}^{\mathcal{R}})_j$  is the direct sum of standard free  $\mathcal{R}_j$ -modules:

$$(h_{i,x}^{\mathcal{R}})_j = \bigoplus_{i \xrightarrow{\alpha} j} h_{\mathcal{R}_\alpha(x)}.$$

It follows that for any  $\mathcal{R}_j$ -module  $X$  one has isomorphisms

$$\mathrm{Hom}_{\mathcal{R}_j}((h_{i,x}^{\mathcal{R}})_j, X) \cong \prod_{i \xrightarrow{\alpha} j} X(\mathcal{R}_\alpha(x)).$$

Thus for any  $\mathcal{R}$ -module  $M$  one has a natural isomorphism

$$\mathrm{Hom}_{\mathcal{R}}(h_{i,x}^{\mathcal{R}}, M) \cong M_i(x).$$

Let now  $k$  be an object of  $\mathbf{I}$  and let  $A$  be an  $\mathcal{R}_k$ -module. We denote by  $k_*(A)$  the  $\mathcal{R}$ -module, whose value at  $i$  is given by

$$(k_*A)_i = \prod_{i \xrightarrow{\alpha} k} \mathcal{R}_\alpha^* A.$$

The  $\alpha$ -component of  $(k_*A)_i$  has an  $\mathcal{R}_i$ -module structure given by restriction of scalars along the ringoid homomorphism  $\mathcal{R}_\alpha : \mathcal{R}_i \rightarrow \mathcal{R}_k$ . Hence  $(k_*A)_i$  is an  $\mathcal{R}_i$ -module and now it is clear that  $k_*A$  is an  $\mathcal{R}$ -module. Moreover the functor  $k_* : \mathcal{R}_k\text{-}\mathbf{mod} \rightarrow \mathcal{R}\text{-}\mathbf{mod}$  is right adjoint to the evaluation functor  $\mathrm{ev}_k : \mathcal{R}\text{-}\mathbf{mod} \rightarrow \mathcal{R}_k\text{-}\mathbf{mod}$ , which is given by  $\mathrm{ev}_k(M) = M_k$ . In particular, if  $A$  is an injective  $\mathcal{R}_k$ -module then  $k_*A$  is an injective  $\mathcal{R}$ -module. Hence the family  $(k_*Q)_{k,Q}$ , is a family of injective cogenerators for the category of  $\mathcal{R}$ -modules. Here  $k$  runs over the set of objects of  $\mathbf{I}$ , and then  $Q$  over the set of injective cogenerators of the category of  $\mathcal{R}_k$ -modules.

From 3.3.1 we also have that for any  $M, N$  in  $\mathcal{R}\text{-}\mathbf{mod}$  we can calculate their Ext groups as Ext groups of the corresponding objects in  $\mathcal{R}[\mathbf{I}]\text{-}\mathbf{mod}$ . We will denote  $\mathrm{Ext}_{\mathcal{R}[\mathbf{I}]}^*(M, N)$  by  $\mathrm{Ext}_{\mathcal{R}\text{-}\mathbf{mod}}^*(M, N)$  and call them *global Hom and Ext groups*. One can also define local Hom and Ext functors, exactly as for the ring valued functors.

**Lemma 3.3.3.** *Let us fix  $i \in \mathbf{I}$  and  $x \in \mathrm{Ob}(\mathcal{R}_i)$ . For any functor  $N : \int_{\mathbf{I}} \mathcal{R} \rightarrow \mathcal{M}$  consider the natural system  $D$  on  $\mathbf{I}$  given by*

$$D_{c \xrightarrow{\varphi} d} := \prod_{i \xrightarrow{\alpha} c} N(d, \mathcal{R}_{\varphi\alpha}(x)).$$

Then

$$H^0(\mathbf{I}; D) = N(i, x).$$

and

$$H^n(\mathbf{I}; D) = 0 \text{ for } n > 0.$$

*Proof.* Consider the *comma category*  $i/\mathbf{I}$  (see e. g. [15]); its objects are arrows  $i \rightarrow j$ , where  $j$  runs over objects the category  $\mathbf{I}$ , and morphism are commutative diagrams

$$\begin{array}{ccc} j & \xrightarrow{\quad} & k \\ & \nwarrow \quad \nearrow & \\ & i & \end{array}$$

One easily checks that

$$C^*(\mathbf{I}; D) \cong C^*(i/\mathbf{I}; T),$$

where  $T : i/\mathbf{I} \rightarrow \mathcal{A}$  is given by

$$T\left(i \xrightarrow{\alpha} c\right) = N(c, \mathcal{R}_\alpha(x)).$$

Hence the cohomology of  $\mathbf{I}$  with coefficients in  $D$  coincides with the cohomology of the category  $i/\mathbf{I}$  with coefficients in the functor  $T$ . Since  $1_i$  is the initial object in the category  $i/\mathbf{I}$  one can use Lemma 2.2.1 to finish the proof.  $\square$

**3.4. Proof of Theorem 3.1.1.** We fix a left  $\mathcal{R}$ -module  $N$ . We claim that for any left  $\mathcal{R}$ -module  $X$  one has an isomorphism:

$$H^0(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(X, N)) \cong \text{Hom}_{\mathcal{R}\text{-mod}}(X, N).$$

Indeed, it follows from the definition of cohomology that  $H^0(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(X, N))$  is isomorphic to the kernel

$$\text{Ker} \left( \prod_{i \in \text{Ob}(\mathbf{I})} \text{Hom}_{\mathcal{R}_i\text{-mod}}(X_i, N_i) \rightarrow \prod_{i \xrightarrow{\alpha} j} \text{Hom}_{\mathcal{R}_i\text{-mod}}(X_i, N_j) \right).$$

Thus  $H^0(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(X, N))$  consists of families  $(f_i : X_i \rightarrow N_i)$  of  $\mathcal{R}_i$ -homomorphisms, such that for any  $\alpha : i \rightarrow j$  the diagram

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & N_i \\ \downarrow X_\alpha & & \downarrow N_\alpha \\ X_j & \xrightarrow{f_j} & N_j \end{array}$$

commutes, and the claim is proved. One observes that the diagram

$$\begin{array}{ccc} & \text{Nat}(\mathbf{I}) & \\ \mathcal{H}om_{\mathcal{R}}(-, N) \nearrow & & \searrow H^0(\mathbf{I}; -) \\ \mathcal{R}\text{-mod}^{\text{op}} & \xrightarrow{\text{Hom}_{\mathcal{R}\text{-mod}}(-, N)} & \mathcal{A} \end{array}$$

commutes and the Theorem is a consequence of the Grothendieck spectral sequence for composite functors. Of course in order to apply the Grothendieck theorem we first have to show that  $H^n(\mathbf{I}; \mathcal{H}om_{\mathcal{R}}(M, N)) = 0$  as soon as  $n > 0$  and  $M$  is projective. To this end we can assume without loss of generality that  $M = h_{i,x}^{\mathcal{R}}$ , for some  $i \in \mathbf{I}$  and  $x \in \mathcal{R}_i$ . In this case

$$\mathcal{H}om_{\mathcal{R}}(M, N)_{c \xrightarrow{\varphi} d} \cong \prod_{i \xrightarrow{\alpha} c} N(d, \mathcal{R}_{\varphi\alpha}(x))$$

and therefore we can use Lemma 3.3.3 to finish the proof.  $\square$



## 4. FINITE PRODUCT THEORIES

**4.1. Basic definitions.** A *finite product theory* (simply theory for us) is a small category with finite products. A morphism of theories is a functor preserving finite products. With these morphisms, theories form a category  $\mathcal{Theories}$ . Let  $\mathcal{C}$  be a category with finite products. A *model* of a theory  $\mathbb{A}$  in the category  $\mathcal{C}$ , also termed a  $\mathcal{C}$ -*valued model* of  $\mathbb{A}$ , or an  $\mathbb{A}$ -*model in*  $\mathcal{C}$ , is a functor  $\mathbb{A} \rightarrow \mathcal{C}$  preserving finite products. Models of  $\mathbb{A}$  in  $\mathcal{C}$  form a category  $\mathbb{A}(\mathcal{C})$ , with natural transformations as morphisms. Models in the category  $\mathcal{Ens}$  of sets will be called simply models, and the category  $\mathbb{A}(\mathcal{Ens})$  will be also denoted by  $\mathbb{A}\text{-}\mathbf{mod}$ . It is known that the category  $\mathbb{A}\text{-}\mathbf{mod}$  is complete and cocomplete for any theory  $\mathbb{A}$ . Moreover the inclusion  $\mathbb{A}\text{-}\mathbf{mod} \hookrightarrow \text{Func}(\mathbb{A}, \mathcal{Ens})$  preserves all limits and has a left adjoint, and the Yoneda embedding  $\mathbb{A}^{\text{op}} \rightarrow \text{Func}(\mathbb{A}, \mathcal{Ens})$  factors through it, i. e. there is a full embedding  $F : \mathbb{A}^{\text{op}} \rightarrow \mathbb{A}\text{-}\mathbf{mod}$ . Models in the image of  $F$  are called *finitely generated free models*, so that  $\mathbb{A}$  is equivalent to the opposite of the category of such models. It is easy to see that the functor  $F$  preserves coproducts, i. e.  $F(X \times Y)$  is a coproduct of  $F(X)$  and  $F(Y)$  in the category of models. A morphism of theories  $f : \mathbb{A} \rightarrow \mathbb{B}$  induces a functor

$$f^* : \mathbb{B}\text{-}\mathbf{mod} \rightarrow \mathbb{A}\text{-}\mathbf{mod},$$

where  $f^*(M) = M \circ f$ . Clearly this functor preserves all limits. Since moreover the categories of models have small generating subcategories (those of free models), by Freyd's Special Adjoint Functor Theorem the functor  $f^*$  has a left adjoint

$$f_! : \mathbb{A}\text{-}\mathbf{mod} \rightarrow \mathbb{B}\text{-}\mathbf{mod}.$$

One can see that the square

$$\begin{array}{ccc} \mathbb{A}^{\text{op}} & \xrightarrow{I_{\mathbb{A}}} & \mathbb{A}\text{-}\mathbf{mod} \\ f^{\text{op}} \downarrow & & \downarrow f_! \\ \mathbb{B}^{\text{op}} & \xrightarrow{I_{\mathbb{B}}} & \mathbb{B}\text{-}\mathbf{mod} \end{array}$$

commutes. See [2] for details.

**4.1.1. Single sorted theories.** Let  $\mathbb{S}^{\text{op}} \hookrightarrow \mathcal{Ens}$  be the full subcategory of  $\mathcal{Ens}$  with the objects  $\mathbf{n} = \{1, \dots, n\}$  for  $n \geq 0$ . Since the category  $\mathbb{S}^{\text{op}}$  has finite coproducts, the category  $\mathbb{S}$ , opposite of the category  $\mathbb{S}^{\text{op}}$  is a theory, which is called the *theory of sets*. To distinguish objects of  $\mathbb{S}$  and  $\mathbb{S}^{\text{op}}$  we redenote objects of  $\mathbb{S}$  by  $X^0 = 1, X^1 = X, X^2, X^3, \dots$ . For any  $1 \leq i \leq n$  we denote by  $x_i : X^n \rightarrow X$  the morphism of  $\mathbb{S}$  corresponding to the map  $\{1\} \rightarrow \mathbf{n}$ , which takes 1 to  $i$ . It is clear that  $\mathbf{n}$  is a coproduct of  $n$  copies of  $\{1\}$  in  $\mathbb{S}^{\text{op}}$ . It follows that  $x_1, \dots, x_n : X^n \rightarrow X$  is a product diagram in  $\mathbb{S}$ . One observes that  $\mathbb{S}(\mathcal{C})$  is equivalent to  $\mathcal{C}$  for any category with finite products  $\mathcal{C}$ . In particular  $\mathbb{S}\text{-}\mathbf{mod}$  is equivalent to the category  $\mathcal{Ens}$ .

A *single sorted theory* is a theory morphism  $\mathbb{S} \rightarrow \mathbb{A}$  which is identity on objects. The full subcategory of  $\mathbb{S}/\mathcal{Theories}$  with single sorted theories as objects will be denoted by  $\mathcal{Th}_1$ . Thus objects of single sorted theories are just natural numbers, which are denoted by  $X^0 = 1, X^1 = X, X^2, X^3, \dots$ . There are projections  $x_1, \dots, x_n$  from  $X^n$  to  $X$ . If  $M$  is a

model of a single sorted theory  $\mathbb{A}$ , then  $M(X)$  is called the *underlying set of  $M$* . It is then equipped with operations  $u_M : M(X)^n \rightarrow M(X)$  for each element  $u$  of  $\text{Hom}_{\mathbb{A}}(X^n, X)$ , satisfying identities prescribed by category structure of  $\mathbb{A}$ . By this reason, elements of  $\text{Hom}_{\mathbb{A}}(X^n, X)$  will be called  *$n$ -ary operations of  $\mathbb{A}$* . Thus for any theory  $\mathbb{A}$ , the category  $\mathbb{A}\text{-}\mathbf{mod}$  is a variety of universal algebras. Conversely, for any variety  $\mathbf{V}$ , the opposite of the category of the algebras freely generated by the sets  $\mathbf{n} = \{1, \dots, n\}$ ,  $n \geq 0$ , is a single sorted theory, whose category of models is equivalent to  $\mathbf{V}$ . For example, theory of groups can be described as follows. Let  $\text{Gr}^{\text{op}}$  be the category with objects  $\mathbf{n}$ ,  $n \geq 0$ . A morphism from  $\mathbf{n}$  to  $\mathbf{m}$  is the same as a homomorphism from the free group on  $\mathbf{n}$  to the free group on  $\mathbf{m}$ . Clearly  $\text{Gr}^{\text{op}}$  is equivalent to the category of finitely generated free groups. Hence it has finite coproducts. Therefore the category  $\text{Gr}$ , the opposite of  $\text{Gr}^{\text{op}}$ , is a theory called the *theory of groups*. There is a unique morphism of theories  $\mathbb{S} \rightarrow \text{Gr}$  which is identity on objects. Thus  $\text{Gr}$  is a single sorted theory. One observes that  $\text{Gr}(\mathcal{C})$  is equivalent to the category of group objects in  $\mathcal{C}$  and in particular  $\text{Gr}\text{-}\mathbf{mod}$  is equivalent to the category of groups.

Similarly there is a full embedding

$$\mathcal{Rings} \rightarrow \mathcal{Th}_1$$

assigning to a ring  $R$  the theory  $\mathbb{M}_R$  of left modules over  $R$ , which is defined as follows. Let  $\mathbb{M}_R$  be the opposite of the full subcategory of the category  $R\text{-}\mathbf{mod}$  of left  $R$ -modules with objects the finitely generated free modules  $0, R, R^2, \dots, R^n, \dots$ ; the evident functor  $\mathbb{S}^{\text{op}} \rightarrow \mathbb{M}_R$  sending  $\mathbf{n}$  to  $R^n$  turns  $\mathbb{M}_R$  into a single-sorted theory whose category of models  $\mathbb{M}_R\text{-}\mathbf{mod}$  is equivalent to  $R\text{-}\mathbf{mod}$ . Explicitly, the module corresponding to a model  $M$  is  $M(R)$ , with addition given by

$$M(R) \times M(R) = M(R^2) \xrightarrow{M(+)} M(R)$$

and action of an  $r \in R$  given by  $M(\cdot r) : M(R) \rightarrow M(R)$ , where  $\cdot r : R \rightarrow R$  is the homomorphism of left  $R$ -modules given by  $x \mapsto xr$ . For any category  $\mathcal{C}$ , the category  $\mathbb{M}_R(\mathcal{C})$  is equivalent to the category of *internal  $R$ -modules* in  $\mathcal{C}$ , i. e. internal abelian groups  $A$  equipped with a unital ring homomorphism  $R \rightarrow \text{End}(A)$ . In particular, we have the theory of abelian groups  $\mathbb{A}b = \mathbb{M}_{\mathbb{Z}}$  such that the category  $\mathbb{A}b(\mathcal{C})$  is equivalent to the category of internal abelian groups in  $\mathcal{C}$ , for any category  $\mathcal{C}$ .

**4.1.2. Multisorted theories.** Let  $I$  be a set and consider the category  $\mathbb{S}^{\text{op}}/I$  of maps  $\mathbf{n} \rightarrow I$  for various sets  $\mathbf{n} = \{1, \dots, n\}$ . Morphisms in  $\mathbb{S}^{\text{op}}/I$  from  $\mathbf{n} \rightarrow I$  to  $\mathbf{m} \rightarrow I$  are commutative diagrams of sets

$$\begin{array}{ccc} \mathbf{n} & \xrightarrow{\quad} & \mathbf{m} \\ & \searrow & \swarrow \\ & I & \end{array}$$

One easily sees that this category has finite coproducts; for example, coproduct of  $f_1 : \mathbf{n}_1 \rightarrow I$  and  $f_2 : \mathbf{n}_2 \rightarrow I$  is  $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : \mathbf{n}_1 \sqcup \mathbf{n}_2 \rightarrow I$ . in fact, the set of objects of  $\mathbb{S}^{\text{op}}/I$  can be identified with the free monoid generated by the set  $I$  in such a way that a word  $i_1 \dots i_n$  represent

the coproduct of the objects  $i_\nu : \mathbf{1} \rightarrow I$ ,  $\nu = 1, \dots, n$ . So any  $f : \mathbf{n} \rightarrow I$  is the coproduct of the objects  $f(1) : \mathbf{1} \rightarrow I$ , ...,  $f(n) : \mathbf{1} \rightarrow I$  in  $\mathbb{S}/I$ . We let  $\mathbb{Fam}_I$  be the opposite of the category  $\mathbb{S}^{\text{op}}/I$ . Then  $\mathbb{Fam}_I$  is a theory called the *theory of  $I$ -indexed families*. To distinguish objects of  $\mathbb{Fam}_I$  and  $\mathbb{S}^{\text{op}}/I$  we denote the object of  $\mathbb{Fam}_I$  corresponding to a map  $f : \mathbf{n} \rightarrow I$  by  $X_f$ . Hence an object of  $\mathbb{Fam}_I$  has the form  $X_{i_1} \times \dots \times X_{i_n}$  for a unique  $n$ -tuple  $(i_1, \dots, i_n) \in I^n$ . It is straightforward to check that the functor

$$(*) \quad \mathbb{Fam}_I(\mathcal{C}) \rightarrow \mathcal{C}^I$$

which assigns to a model  $M : \mathbb{Fam}_I \rightarrow \mathcal{C}$  the family  $M(X_i)_{i \in I}$  is an equivalence.

For a set  $I$ , an  *$I$ -sorted theory* is a theory morphism  $\mathbb{Fam}_I \rightarrow \mathbb{A}$  which is identity on objects. The full subcategory of  $\mathbb{Fam}_I/\mathcal{T}heories$  with  $I$ -sorted theories as objects will be denoted by  $\mathcal{T}h_I$ .

Although  $I$ -sorted theories appear to be of very special kind, one has

**Proposition 4.1.1.** *For any theory  $\mathbb{A}$  there is a set  $I$  and an  $I$ -sorted theory  $\mathbb{Fam}_I \rightarrow \tilde{\mathbb{A}}$  such that the category  $\tilde{\mathbb{A}}$  is equivalent to  $\mathbb{A}$ .*

*Proof.* Let  $I$  be the set  $\text{Ob}(\mathbb{A})$  of objects of  $\mathbb{A}$ . We then are forced to take for the set of objects of  $\tilde{\mathbb{A}}$  the free monoid  $\sum_{n \geq 0} \text{Ob}(\mathbb{A})^n$  on  $I$ . There is an obvious map from this monoid to the set of objects of  $\mathbb{A}$ ,  $\Pi : \text{Ob}(\tilde{\mathbb{A}}) \rightarrow \text{Ob}(\mathbb{A})$  which assigns to an  $n$ -tuple  $(X_1, \dots, X_n)$  of objects of  $\mathbb{A}$  its product  $X_1 \times \dots \times X_n$  in  $\mathbb{A}$ . We then simply define

$$\text{Hom}_{\tilde{\mathbb{A}}}((X_1, \dots, X_n), (Y_1, \dots, Y_m)) = \text{Hom}_{\mathbb{A}}(\Pi(X_1, \dots, X_n), \Pi(Y_1, \dots, Y_m)).$$

This clearly defines the category  $\tilde{\mathbb{A}}$  with the same objects as  $\mathbb{Fam}_{\text{Ob}(\mathbb{A})}$  and a functor  $\tilde{\mathbb{A}} \rightarrow \mathbb{A}$  which is full and faithful and surjective on objects, i. e. it is an equivalence. Moreover by (\*) above, models of  $\mathbb{Fam}_{\text{Ob}(\mathbb{A})}$  in a category with finite products  $\mathcal{C}$  are families  $(C_X)_{X \in \text{Ob}(\mathbb{A})}$  of objects of  $\mathcal{C}$ , so the tautological family  $(X)_{X \in \text{Ob}(\mathbb{A})}$  gives a finite product preserving functor  $\mathbb{Fam}_{\text{Ob}(\mathbb{A})} \rightarrow \mathbb{A}$ . It is then obvious that this functor lifts to a functor  $\mathbb{Fam}_{\text{Ob}(\mathbb{A})} \rightarrow \tilde{\mathbb{A}}$  which is identity on objects.  $\square$

A model of an  $I$ -sorted theory  $\mathbb{Fam}_I \rightarrow \mathbb{A}$  is just an  $\mathbb{A}$ -model. For such a model  $\mathbb{A} \rightarrow \mathcal{C}$  in a category  $\mathcal{C}$  its *underlying family* is the object of  $\mathcal{C}^I$  corresponding to the composite  $\mathbb{Fam}_I \rightarrow \mathbb{A} \rightarrow \mathcal{C}$ . When safe, we will denote images of morphisms  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$  of  $\mathbb{A}$  under a model  $\mathbb{A} \rightarrow \mathcal{C}$  by  $\omega$  again. Thus intuitively, models  $M$  of an  $I$ -sorted theory  $\mathbb{Fam}_I \rightarrow \mathbb{A}$  in categories with finite products  $\mathcal{C}$  are  $I$ -tuples of objects  $(C_i)_{i \in I}$ ,  $C_i = M(X_i)$ , equipped with additional structure, namely various operations of the form

$$\omega : C_{i_1} \times \dots \times C_{i_n} \rightarrow C_i$$

corresponding to morphisms  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$  in  $\mathbb{A}$ . These operations must further satisfy various identities expressing the fact that  $M$  is a product preserving functor. In detail, this amounts to the following:

- the morphisms corresponding to the projections  $\pi_1 : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_{i_1}$ , ...,  $\pi_n : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_{i_n}$  must be product projections themselves;

- for morphisms  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$ ,  $\omega' : X_{i'_1} \times \dots \times X_{i'_m} \rightarrow X_i$  and  $\omega_1 : X_{i'_1} \times \dots \times X_{i'_m} \rightarrow X_{i_1}$ , ...,  $\omega_n : X_{i'_1} \times \dots \times X_{i'_m} \rightarrow X_{i_n}$  in  $\mathbb{A}$  with  $\omega(\omega_1, \dots, \omega_n) = \omega'$ , the diagram

$$\begin{array}{ccc}
 & C_{i_1} \times \dots \times C_{i_n} & \\
 \omega \swarrow & & \nwarrow (\omega_1, \dots, \omega_n) \\
 C_i & \xleftarrow{\omega'} C_{i'_1} \times \dots \times C_{i'_m} &
 \end{array}$$

must commute.

The “substrate” underlying the structure of an  $I$ -sorted theory is a family of sets of the form  $(S_{(i_1, \dots, i_n), i})_{(i_1, \dots, i_n) \in I^n, i \in I}$  for  $n = 0, 1, \dots$ , namely, the sets  $\text{Hom}_{\mathbb{A}}(X_{i_1} \times \dots \times X_{i_n}, X_i)$ . We thus have a forgetful functor

$$U : \mathcal{T}_I \rightarrow \prod_{n \geq 0} \mathcal{E}ns^{I^n \times I}.$$

It is proved in [10] that this functor admits a left adjoint  $F$ . Theories in the image of this left adjoint are *free* theories. It is more or less obvious that the adjunction counits  $FU\mathbb{A} \rightarrow \mathbb{A}$  are all full functors, so that in particular one has

**Proposition 4.1.2.** *For any theory  $\mathbb{A}$  there exists a morphism  $\mathbb{F} \rightarrow \mathbb{A}$  from a free theory to  $\mathbb{A}$  which is a full functor.*

□

Moreover, since every componentwise surjective map in  $\prod_{n \geq 0} \mathcal{E}ns^{I^n \times I}$  admits a section, it follows

**Proposition 4.1.3.** *Let  $P : \mathbb{A} \rightarrow \mathbb{F}$  be a morphism in  $\mathcal{T}_I$  which is a full functor. If  $\mathbb{F}$  is a free theory, then  $P$  has a section, i. e. there is a morphism  $S : \mathbb{F} \rightarrow \mathbb{A}$  in  $\mathcal{T}_I$  with  $PS = 1$ .*

□

There is a functor  $\mathcal{R}ingoids \rightarrow \mathcal{T}heories$ . It assigns to a ringoid  $\mathcal{R}$  the theory  $\mathbb{M}_{\mathcal{R}}$  of  $\mathcal{R}$ -modules.  $\mathbb{M}_{\mathcal{R}}$  is the additive category freely generated by  $\mathcal{R}$ , i. e. it is an additive category equipped with a homomorphism of ringoids  $I_{\mathcal{R}} : \mathcal{R} \rightarrow \mathbb{M}_{\mathcal{R}}$  which has the following universal property: for any additive category  $\mathcal{A}$ , precomposition with  $I_{\mathcal{R}}$  induces an equivalence of categories

$$\text{Add}(\mathbb{M}_{\mathcal{R}}, \mathcal{A}) \cong \text{Hom}_{\mathcal{R}ingoids}(\mathcal{R}, \mathcal{A}).$$

There exists an explicit description of  $\mathbb{M}_{\mathcal{R}}$  as the category of *matrices* over  $\mathcal{R}$ :  $\mathbb{M}_{\mathcal{R}}$  can be chosen to be an  $\text{Ob}(\mathcal{R})$ -sorted theory, so that its objects are finite families of objects of  $\mathcal{R}$ , pictured as  $a_1 \oplus \dots \oplus a_n$ , for any  $a_1, \dots, a_n \in \mathcal{R}$ ,  $n \geq 0$ . Moreover  $\text{Hom}_{\mathbb{M}_{\mathcal{R}}}(a_1 \oplus \dots \oplus a_n, b_1 \oplus \dots \oplus b_m)$  is defined as

$$\prod_{\substack{i=1, \dots, m \\ j=1, \dots, n}} \text{Hom}_{\mathcal{R}}(a_j, b_i),$$

with composition defined via matrix multiplication, i. e.  $(f \circ g)_{ik} = \sum_j f_{ij} g_{jk}$  for  $f_{ij} : b_j \rightarrow c_i$ ,  $g_{jk} : a_k \rightarrow b_j$ .

4.1.3. *Tensor product of theories.* In [10] one finds another useful construction on theories:

**Proposition 4.1.4.** *For an  $I$ -sorted theory  $\mathbb{A}$  and a  $J$ -sorted one,  $\mathbb{B}$ , there is an  $I \times J$ -sorted theory  $\mathbb{A} \otimes \mathbb{B}$ , called the Kronecker product of  $\mathbb{A}$  and  $\mathbb{B}$  such that for any category with products  $\mathcal{C}$  one has an equivalence of categories*

$$\mathbb{A}(\mathbb{B}(\mathcal{C})) \simeq (\mathbb{A} \otimes \mathbb{B})(\mathcal{C}).$$

□

4.1.4. *Integrals and cointegrals.* There is a general form of the constructions from 3.3. This is a variation on the *Grothendieck construction, or integral*, which we briefly recall.

Suppose given a functor  $\mathbf{F} : \mathbf{I} \rightarrow \mathbf{CAT}$  from a small category  $\mathbf{I}$  to the category of categories, denoted  $(\varphi : i \rightarrow j) \mapsto (F_\varphi : \mathbf{F}_i \rightarrow \mathbf{F}_j)$ . Then the Grothendieck construction  $\int_{\mathbf{I}} \mathbf{F}$  of  $\mathbf{F}$  is defined as the lax colimit of  $\mathbf{F}$ . Explicitly, it is a category with objects of the form  $(i, X)$ , with  $i \in \text{Ob}(\mathbf{I})$  and  $X \in \text{Ob}(\mathbf{F}_i)$ ; morphisms  $(i, X) \rightarrow (i', X')$  are defined to be pairs  $(\varphi, f)$ , with  $\varphi : i \rightarrow i'$  and  $f : F_\varphi(X) \rightarrow X'$ . Identity morphism for  $(i, X)$  is  $(\text{id}_i, \text{id}_X)$ , and composition of  $(\varphi' : i' \rightarrow i'', f' : F_{\varphi'}(X') \rightarrow X'')$  with  $(\varphi, f)$  as above is defined to be the pair  $(\varphi'\varphi, f'F_{\varphi'}(f))$ . There is a canonical functor  $P_{\mathbf{F}} : \int_{\mathbf{I}} \mathbf{F} \rightarrow \mathbf{I}$  given by projection onto the first coordinate, i. e. sending  $(i, X)$  to  $i$  and  $(\varphi, f)$  to  $\varphi$ .

We will also need the less known *lax limit, or cointegral* of a functor like  $\mathbf{F}$ , which we will denote by  $\int^{\mathbf{I}} \mathbf{F}$  (cf. e. g. [1, VI 7], [12, 5.2.3] or [11, I,7.12]). This is equal to the category of sections of the functor  $P_{\mathbf{F}}$  above. Thus its objects can be identified with pairs of families

$$\left( (X_i)_{i \in \text{Ob}(\mathbf{I})}, (F_\varphi(X_i) \xrightarrow{f_\varphi} X_{i'})_{(i \xrightarrow{\varphi} i') \in \text{Mor}(\mathbf{I})} \right)$$

satisfying  $f_{\text{id}_i} = \text{id}_{X_i}$  for any  $i \in \text{Ob}(\mathbf{I})$  and  $f_{\varphi'\varphi} = f_{\varphi'} F_{\varphi'}(f_\varphi)$  for any  $\varphi : i \rightarrow i'$ ,  $\varphi' : i' \rightarrow i''$ . Whereas morphisms  $(X_*, f_*) \rightarrow (Y_*, g_*)$  are families  $\Phi_i : X_i \rightarrow Y_i$  making all the diagrams

$$\begin{array}{ccc} F_\varphi(X_i) & \xrightarrow{F_\varphi(\Phi_i)} & F_\varphi(Y_i) \\ f_\varphi \downarrow & & \downarrow g_\varphi \\ X_{i'} & \xrightarrow{\Phi_{i'}} & Y_{i'} \end{array}$$

commute.

We will need the following

**Lemma 4.1.5.** *A cointegral of theories is a theory. More precisely, suppose given a functor  $\mathbf{F} : \mathbf{I} \rightarrow \mathbf{CAT}$  such that each category  $\mathbf{F}_i$  has finite products and each functor  $F_\varphi : \mathbf{F}_i \rightarrow \mathbf{F}_{i'}$  preserves them. Then both categories  $\int^{\mathbf{I}} \mathbf{F}$  and  $(\int^{\mathbf{I}} (\mathbf{F}^{\text{op}}))^{\text{op}}$  also have finite products which are computed componentwise, that is, for example, for two objects  $(f_\varphi :$*

$X_i \rightarrow F_\varphi(X_{i'})_{\varphi:i \rightarrow i'}$  and  $(g_\varphi : Y_i \rightarrow F_\varphi(Y_{i'}))_{\varphi:i \rightarrow i'}$  of  $(\int^{\mathbf{I}}(\mathbf{F}^{\text{op}}))^{\text{op}}$  their product is given by the family of the composites

$$X_i \times Y_i \xrightarrow{f_\varphi \times g_\varphi} F_\varphi(X_{i'}) \times F_\varphi(Y_{i'}) \xrightarrow{\cong} F_\varphi(X_{i'} \times Y_{i'}).$$

Proof is straightforward.  $\square$

Here is an example when such lax limit appears in our context:

**Proposition 4.1.6.** *Let  $\mathcal{R} : \mathbf{I} \rightarrow \mathcal{R}\text{ingoids}$  be a ringoid valued functor on a small category  $\mathbf{I}$ . Then the category  $\mathcal{R}\text{-mod}$  described in 3.3 is equivalent to  $(\int^{\mathbf{I}^{\text{op}}} \mathbf{F})^{\text{op}}$  for the functor  $\mathbf{F} : \mathbf{I}^{\text{op}} \rightarrow \mathbf{CAT}$  sending  $i \in \text{Ob}(\mathbf{I})$  to the category  $\mathcal{R}_i\text{-mod}^{\text{op}}$  of modules over the ringoid  $\mathcal{R}_i$  and  $\varphi : i \rightarrow j$  — to the “restriction of scalars” functor  $\mathcal{R}_\varphi^* : \mathcal{R}_j\text{-mod} \rightarrow \mathcal{R}_i\text{-mod}$  induced by the ringoid homomorphism  $\mathcal{R}_\varphi : \mathcal{R}_i \rightarrow \mathcal{R}_j$ .*

*Proof.* This follows straightforwardly from the definition of the category  $\mathcal{R}\text{-mod}$  in 3.3.  $\square$

A construction similar to that of 3.3 can be performed with theories too.

Let  $\mathbb{A} : \mathbf{I} \rightarrow \mathcal{T}\text{heories}$  be a functor from a small category  $\mathbf{I}$  to theories. Define a *model* of  $\mathbb{A}$  to be a collection  $(M_i)_{i \in \text{Ob}(\mathbf{I})}$  of  $\mathbb{A}_i$ -models, one for each  $i \in \text{Ob}(\mathbf{I})$ , together with a collection of morphisms  $M_\varphi : M_i \rightarrow \mathbb{A}_\varphi^* M_{i'}$ , one for each  $\varphi : i \rightarrow i'$  in  $\text{Mor}(\mathbf{I})$ , such that  $M_{\text{id}_i} = \text{id}_{M_i}$  and  $M_{\varphi'\varphi} = \mathbb{A}_{\varphi'}^*(M_{\varphi'})M_\varphi$  for any  $\varphi : i \rightarrow i'$ ,  $\varphi' : i' \rightarrow i''$ . Thus also straightforwardly one has

**Proposition 4.1.7.** *For any functor  $\mathbb{A} : \mathbf{I} \rightarrow \mathcal{T}\text{heories}$ , the category of  $\mathbb{A}$ -models is equivalent to  $(\int^{\mathbf{I}^{\text{op}}} F)^{\text{op}}$ , where  $F : \mathbf{I} \rightarrow \mathbf{CAT}$  assigns the category  $\mathbb{A}_i\text{-mod}$  to  $i \in \text{Ob}(\mathbf{I})$  and the functor  $\mathbb{A}_\varphi^*$  to  $\varphi : i \rightarrow i'$ .*

$\square$

4.1.5. *Comma category as models.* As an application of previous discussion we prove that the comma category of a category of models of a theory is still a category of models for a theory.

**Proposition 4.1.8.** *For an  $I$ -sorted theory  $\mathbb{A}$  and any model  $M$  in  $\mathbb{A}\text{-mod}$ , the category  $\int_{\mathbb{A}} M$  is a  $(\coprod_{i \in I} M_i)$ -sorted theory and moreover the comma category  $\mathbb{A}\text{-mod}/M$  is equivalent to the category of models  $(\int_{\mathbb{A}} M)\text{-mod}$ .*

*Proof.* Any object  $N$  of  $\mathbb{A}\text{-mod}$  equipped with a morphism  $f : N \rightarrow M$  can be considered as a collection of sets

$$(N_x = f_A^{-1}(x) \subseteq N(A))_{x \in \coprod_{A \in \text{Ob}(\mathbb{A})} M(A)}$$

and maps  $N_{x_1} \times \dots \times N_{x_n} \rightarrow N_{\omega(x_1, \dots, x_n)}$ , for all  $(x_1, \dots, x_n) \in M(X_{i_1}) \times \dots \times M(X_{i_n})$  and  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$  in  $\mathbb{A}$ , fitting into certain commutative diagrams.

Then regarding  $M$  as an object of  $\mathcal{E}_{ns}^{\mathbb{A}}$ , and defining  $N(x) = N_{M(p_1)x} \times \dots \times N_{M(p_n)x}$ , for  $x \in M(X_{i_1} \times \dots \times X_{i_n})$ , we can consider the above data as a functor  $\tilde{N} : \int_{\mathbb{A}} M \rightarrow \mathcal{E}_{ns}$ , which sends the object  $x \in M(X_{i_1} \times \dots \times X_{i_n})$  of the latter category to the product of the objects  $\tilde{N}(X_{i_\nu})$ ,  $\nu = 1, \dots, n$ . Now the proof follows from the subsequent lemma.  $\square$

**Lemma 4.1.9.** *A functor  $M : \mathbb{A} \rightarrow \mathcal{E}_{ns}$  preserves finite products if and only if the category  $\int_{\mathbb{A}} M$  has finite products and the canonical functor  $P : \int_{\mathbb{A}} M \rightarrow \mathbb{A}$ , sending  $m \in M(X)$  to  $X$ , preserves them.*

*Proof.* Let us first recall that functors of the form  $P : \int_{\mathbb{A}} M \rightarrow \mathbb{A}$  for any functor  $M : \mathbb{A} \rightarrow \mathcal{E}_{ns}$  are characterized by a property called *discrete opfibration*:

for any  $x \in \int_{\mathbb{A}} M$  and any  $\varphi : Px \rightarrow a$ , there is a unique  $\psi : x \rightarrow y$  with  $P\psi = \varphi$ .

Using this property it is easy to prove that a pullback of a product preserving discrete fibration between categories with products along a product preserving functor is again a product preserving functor between categories with products.

The “only if” part then follows because of the following pullback diagram in the category of categories

$$\begin{array}{ccc} \int_{\mathbb{A}} M & \longrightarrow & \mathcal{E}_{ns} \bullet \\ \downarrow P & & \downarrow U \\ \mathbb{A} & \xrightarrow{M} & \mathcal{E}_{ns} \end{array}$$

in which  $\mathcal{E}_{ns} \bullet$  denotes the category of pointed sets and  $U$  the forgetful functor: since the latter is a discrete opfibration and preserves products, it follows that  $\int_{\mathbb{A}} M$  will have and  $P : \int_{\mathbb{A}} M \rightarrow \mathbb{A}$  preserve them too.

For the “if” part, we again use the discrete fibration property to prove

- a)  $M(1)$  has single element: the particular case of the above discrete opfibration condition with  $Px = a = 1$  implies that for any  $x \in P^{-1}(1)$  one has  $\left(x \xrightarrow{\text{id}_x} x\right) = \left(x \xrightarrow{!_x} 1\right)$ , since  $P(\text{id}_x) = P(!_x) = \text{id}_1$ .
- b)  $M(a_1 \times a_2) \xrightarrow{(M\pi_1, M\pi_2)} Ma_1 \times Ma_2$  is bijective: this follows from another two particular cases of the discrete opfibration condition — with  $x = x_1 \times x_2$  for some  $x_i \in P^{-1}(a_i)$  and  $\varphi = \pi_i$ ,  $i = 1, 2$ ; indeed these cases give that there are unique  $\psi_i$  starting out of  $x$  with  $P(\psi_i) = \pi_i$ , hence  $x$  is a unique element of  $M(a_1 \times a_2)$  satisfying  $M\pi_i(x) = x_i$ ,  $i = 1, 2$ .

□

**Corollary 4.1.10.** *For any theory  $\mathbb{A}$  and any functor  $M : \mathbf{I} \rightarrow \mathbb{A}\text{-mod}$ , there is an equivalence*

$$\mathbb{A}\text{-mod}^{\mathbf{I}}/M \simeq \mathbb{A}/M\text{-mod},$$

where  $\mathbb{A}/M : \mathbf{I} \rightarrow \mathcal{T}heories$  is the functor given by  $i \mapsto \int_{\mathbb{A}} M_i$ .

*Proof.* An object of  $\mathbb{A}\text{-}\mathbf{mod}^{\mathbf{I}}/M$  consists of homomorphisms of  $\mathbb{A}$ -models  $p_i : N_i \rightarrow M_i$ ,  $i \in \text{Ob}(\mathbf{I})$ , and  $N_\varphi : N_i \rightarrow N_{i'}$ ,  $\varphi : i \rightarrow i'$ , such that all squares

$$\begin{array}{ccc} N_i & \xrightarrow{N_\varphi} & N_{i'} \\ p_i \downarrow & & \downarrow p_{i'} \\ M_i & \xrightarrow{M_\varphi} & M_{i'} \end{array}$$

commute and moreover  $N_{\text{id}_i} = \text{id}_{N_i}$ ,  $N_{\varphi' \circ \varphi} = N_{\varphi'} \circ N_\varphi$  for all  $i$  and all composable pairs  $\varphi', \varphi$ .

It is then clear that such data can be equivalently figured out as a collection of objects  $(N_i, p_i) \in \mathbb{A}\text{-}\mathbf{mod}/M_i$ ,  $i \in \text{Ob}(\mathbf{I})$ , together with a collection of morphisms  $(N_i, p_i) \rightarrow M_\varphi^*(N_{i'}, p_{i'})$ , for  $\varphi : i \rightarrow i'$  in  $\mathbf{I}$  which satisfy exactly the conditions determining an object of  $(f^{\mathbf{I}^{\text{op}}}(\mathbb{A}/M(\_))^{\text{op}})^{\text{op}}$ , i. e., by definition, of  $\mathbb{A}/M\text{-}\mathbf{mod}$ . It is straightforward to check that this correspondence also carries over to morphisms.  $\square$

#### 4.2. Enveloping ringoids.

**Proposition 4.2.1.** *For any  $I$ -sorted theory  $\mathbb{A}$  there exists a ringoid  $U(\mathbb{A})$ , depending functorially on  $\mathbb{A}$ , such that  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod})$  is equivalent to the category of  $U(\mathbb{A})$ -modules.*

*Proof.* The key observation here is that in the presence of an abelian group structure any operation like  $\omega : X_1 \times \dots \times X_n \rightarrow X$  must be an abelian group homomorphism, hence have the form  $\omega(x_1, \dots, x_n) = \omega_1(x_1) + \dots + \omega_n(x_n)$  for some unary operations  $\omega_i : X_i \rightarrow X$ .

Let the set of objects of  $U(\mathbb{A})$  be  $I$ , and present morphisms of  $U(\mathbb{A})$  by generators and relations as follows. For each  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$  in  $\mathbb{A}$  we pick  $n$  generators  $\langle \omega, 1 \rangle : X_{i_1} \rightarrow X_i$ , ...,  $\langle \omega, n \rangle : X_{i_n} \rightarrow X_i$ . And for each such  $\omega$  and any  $\omega_1 : X_{i'_1} \times \dots \times X_{i'_m} \rightarrow X_{i_1}$ , ...,  $\omega_n : X_{i'_1} \times \dots \times X_{i'_m} \rightarrow X_{i_n}$  we impose the relations

$$\langle \omega(\omega_1, \dots, \omega_n), \mu \rangle = \sum_{\nu=1}^n \langle \omega, \nu \rangle \circ \langle \omega_\nu, \mu \rangle$$

for  $\mu = 1, \dots, m$ . So a  $U(\mathbb{A})$ -module is a collection of abelian groups  $(A_i)_{i \in I}$  and homomorphisms  $\langle \omega, \nu \rangle : A_{i_\nu} \rightarrow A_i$ ,  $\omega \in \text{Hom}_{\mathbb{A}}(X_{i_1} \times \dots \times X_{i_n}, X_i)$ ,  $\nu = 1, \dots, n$  satisfying the above relations. Then from any such module we obtain an object of  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod})$  by defining

$$\omega(a_1, \dots, a_n) = \sum_{\nu=1}^n \langle \omega, \nu \rangle a_\nu$$

for  $\omega$  as above and  $(a_1, \dots, a_n) \in A_{i_1} \times \dots \times A_{i_n}$ . Conversely, if  $(A_i)_{i \in I}$  is given the structure of an object from  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod})$ , then we define

$$\langle \omega, \nu \rangle a = \omega(0, \dots, 0, \underset{\nu\text{-th position}}{a}, 0, \dots, 0).$$

It is easy to see that these procedures determine mutually inverse equivalences between the category of  $U(\mathbb{A})$ -modules and  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod})$ .  $\square$

One then has



**Corollary 4.2.2.** *For a theory  $\mathbb{A}$ , the following conditions are equivalent:*

- i)  $\mathbb{A}$  is isomorphic to  $\mathbb{M}_{\mathcal{R}}$  for some ringoid  $\mathcal{R}$ ;
- ii)  $\mathbb{A}\text{-}\mathbf{mod}$  is an additive category;
- iii) the canonical homomorphism of theories  $\mathbb{A} \rightarrow \mathbb{A}\mathbf{b} \otimes \mathbb{A}$  is an isomorphism;
- iv)  $\mathbb{A}$  is isomorphic to  $\mathbb{A}\mathbf{b} \otimes \mathbb{B}$  for some theory  $\mathbb{B}$ .

*Proof.* Implication i) $\Rightarrow$ ii) is clear, ii)  $\iff$  iii) follows from the fact that a category  $\mathcal{A}$  with finite products is additive iff the forgetful functor  $\mathbb{A}\mathbf{b}(\mathcal{A}) \rightarrow \mathcal{A}$  is an equivalence, and iii) $\Rightarrow$ iv) is trivial. Finally iv) $\Rightarrow$ i) follows from the above proposition.  $\square$

**Corollary 4.2.3.** *For any model  $M$  of a theory  $\mathbb{A}$ , there exists a ringoid  $\mathcal{U}(M)$ , the enveloping ringoid of  $M$ , depending functorially on  $M$ , such that the category  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod}/M)$  is equivalent to the category of  $\mathcal{U}(M)$ -modules.*

*Proof.* Of course this is just a particular case of the previous proposition in view of 4.1.8. Let us, however, give explicit presentation of  $\mathcal{U}(M) = U(\int_{\mathbb{A}} M)$  in this case, assuming for simplicity that  $\mathbb{A}$  is an  $I$ -sorted theory. The set of objects of  $\mathcal{U}(M)$  is then  $\coprod_{i \in I} M(X_i)$ , and the morphisms are generated by ones of the form  $\langle \omega, x_1, \dots, x_n, \nu \rangle : x_{i_\nu} \rightarrow \omega(x_1, \dots, x_n)$ , for each  $\omega \in \text{Hom}_{\mathbb{A}}(X_{i_1} \times \dots \times X_{i_n}, X_i)$ ,  $(x_1, \dots, x_n) \in M(X_{i_1}) \times \dots \times M(X_{i_n})$  and  $\nu \in \{1, \dots, n\}$ . The defining relations are indexed by data  $\omega \in \text{Hom}_{\mathbb{A}}(X_{i_1} \times \dots \times X_{i_n}, X_i)$ ,  $\omega_1 \in \text{Hom}_{\mathbb{A}}(X_{i'_1} \times \dots \times X_{i'_m}, X_{i_1})$ ,  $\dots$ ,  $\omega_n \in \text{Hom}_{\mathbb{A}}(X_{i'_1} \times \dots \times X_{i'_m}, X_{i_n})$ ,  $(x_1, \dots, x_m) \in M(X_{i'_1}) \times \dots \times M(X_{i'_m})$ , and  $\mu \in \{1, \dots, m\}$  and have the form

$$\langle \omega(\omega_1, \dots, \omega_n), x_1, \dots, x_m, \mu \rangle = \sum_{\nu=1}^n \langle \omega, \omega_1(x_1, \dots, x_m), \dots, \omega_n(x_1, \dots, x_m), \nu \rangle \circ \langle \omega_\nu, x_1, \dots, x_m, \mu \rangle.$$

Once again, functoriality is obvious from this presentation.  $\square$

Occasionally we will write  $\mathcal{U}_{\mathbb{A}}(M)$  to make explicit dependence on  $\mathbb{A}$ . This construction is known under various names in the literature — see e. g. [3] or [20]. We will also need a generalization of this fact to functors, which requires the following

**Lemma 4.2.4.** *Given a theory  $\mathbb{A}$  and a functor  $\mathbf{F} : I \rightarrow \mathbf{CAT}$ , there is an equivalence*

$$\mathbb{A} \left( \int^I \mathbf{F} \right) \simeq \int^I \mathbb{A}(\mathbf{F}_-),$$

where  $\mathbb{A}(\mathbf{F}_-) : I \rightarrow \mathbf{CAT}$  is given by  $i \mapsto \mathbb{A}(\mathbf{F}_i)$ .

*Proof.* It is easy to see that for any category  $\mathbb{A}$  whatsoever there is an equivalence

$$\text{Func}(\mathbb{A}, \int^I \mathbf{F}) \simeq \int^I \mathbf{F}^{\mathbb{A}},$$

where  $\mathbf{F}^{\mathbb{A}} : I \rightarrow \mathbf{CAT}$  is given by  $i \mapsto \text{Func}(\mathbb{A}, \mathbf{F}_i)$ . On the other hand we know by 4.1.5 that products in  $\int^I \mathbf{F}$  are computed componentwise; this implies easily that an object of  $\int^I \mathbf{F}^{\mathbb{A}}$  like  $f_\varphi : M_i \rightarrow F_\varphi M'_i$ , with  $M_i : \mathbb{A} \rightarrow \mathbf{F}_i$ , etc. corresponds to an  $\mathbb{A}$ -model in  $\int^I \mathbf{F}$  iff each  $M_i$  is an  $\mathbb{A}$ -model in  $\mathbf{F}_i$ .  $\square$

**Proposition 4.2.5.** *For any theory  $\mathbb{A}$  and any functor  $M : \mathbf{I} \rightarrow \mathbb{A}\text{-}\mathbf{mod}$ , there is an equivalence of categories*

$$\mathbb{A}b(\mathrm{Func}(\mathbf{I}, \mathbb{A}\text{-}\mathbf{mod})/M) \simeq \mathcal{U}(M)\text{-}\mathbf{mod},$$

where  $\mathcal{U}(M) : \mathbf{I} \rightarrow \mathcal{R}\text{ingoids}$  is the functor defined by  $i \mapsto \mathcal{U}(M_i)$ .

*Proof.* From 4.1.10, there is an equivalence

$$\mathbb{A}b(\mathrm{Func}(\mathbf{I}, \mathbb{A}\text{-}\mathbf{mod})/M) \simeq \mathbb{A}b(\mathbb{A}/M\text{-}\mathbf{mod});$$

but by definition

$$\mathbb{A}/M\text{-}\mathbf{mod} = \left( \int^{\mathbf{I}^{\mathrm{op}}} (\mathbb{A}\text{-}\mathbf{mod}/M_-)^{\mathrm{op}} \right)^{\mathrm{op}},$$

Hence by 4.2.4, there is an equivalence

$$\mathbb{A}b(\mathbb{A}/M\text{-}\mathbf{mod}) = \left( \int^{\mathbf{I}^{\mathrm{op}}} \mathbf{F} \right)^{\mathrm{op}},$$

where  $\mathbf{F} : \mathbf{I} \rightarrow \mathbf{CAT}$  is the functor given by  $i \mapsto \mathbb{A}b(\mathbb{A}\text{-}\mathbf{mod}/M_i)$ . Now by 4.2.1,  $\mathbf{F}$  is isomorphic to the functor  $\mathcal{U}(M_-)\text{-}\mathbf{mod} : \mathbf{I} \rightarrow \mathbf{CAT}$  given by  $i \mapsto \mathcal{U}(M_i)\text{-}\mathbf{mod}$ ; and we have proved in 4.1.6 that there is an equivalence

$$\left( \int^{\mathbf{I}^{\mathrm{op}}} \mathcal{U}(M_-)\text{-}\mathbf{mod} \right)^{\mathrm{op}} \simeq \mathcal{U}(M)\text{-}\mathbf{mod}.$$

□

Given a theory  $\mathbb{A}$ , its model  $M \in \mathbb{A}\text{-}\mathbf{mod}$ , and an object  $p : A \rightarrow M$  of the category  $\mathbb{A}b(\mathbb{A}\text{-}\mathbf{mod}/M) \simeq \mathcal{U}_{\mathbb{A}}(M)\text{-}\mathbf{mod}$ , we will denote by  $\mathrm{Der}(M; A)$  the abelian group of all sections of  $A \rightarrow M$ , i. e. the set of all morphisms  $s : M \rightarrow A$  of  $\mathbb{A}$ -models with  $ps = 1_M$ . Elements of  $\mathrm{Der}(M; A)$  will be called *derivations* of  $M$  in  $A$ .  $\mathrm{Der}(M; A)$  is contravariantly functorial in  $M$ , in the following sense. For a morphism  $f : M' \rightarrow M$  of models we get the induced homomorphism  $f^* : \mathrm{Der}(M; A) \rightarrow \mathrm{Der}(M'; f^*A)$ , where  $f^*A$  denotes the pullback of  $p : A \rightarrow M$  along  $f$ . Equivalently, one might interpret  $\mathrm{Der}(M'; f^*A)$  as the abelian group of all  $\mathbb{A}$ -model morphisms  $M' \rightarrow A$  over  $M$ , i. e. fitting in the commutative diagram

$$(\ddagger) \quad \begin{array}{ccc} & & A \\ & \nearrow f & \downarrow p \\ M' & \xrightarrow{\quad} & M \end{array}.$$

Clearly also  $\mathrm{Der}(M; A)$  is covariantly functorial in  $A$  and so defines a functor  $\mathrm{Der}(M; \_)$  on  $\mathcal{U}_{\mathbb{A}}(M)\text{-}\mathbf{mod}$ . We then have

**Proposition 4.2.6.** *The functor  $\mathrm{Der}(M; \_)$  is representable. That is, there exists an  $\mathcal{U}_{\mathbb{A}}(M)$ -module  $\Omega_M^1$  with natural isomorphism  $\mathrm{Der}(M; A) \cong \mathrm{Hom}_{\mathcal{U}_{\mathbb{A}}(M)}(\Omega_M^1, A)$  for all  $A$ . Moreover  $\Omega^1$  depends functorially on  $M$ . When  $M$  is a finitely generated free  $\mathbb{A}$ -model, then  $\Omega_M^1$  is a projective object of  $\mathcal{U}(M)\text{-}\mathbf{mod}$ .*

*Proof.* Following the equivalence from 4.2.3, for an  $\mathcal{U}(M)$ -module  $A$  the corresponding object of  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod}/M)$  is the  $\mathbb{A}$ -model with  $X_i \mapsto \coprod_{x \in M(X_i)} A(x)$ , with the  $\mathbb{A}$ -model structure assigning to  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$  the operation

$$\omega : \coprod_{x_1 \in M(X_{i_1})} A(x_1) \times \dots \times \coprod_{x_n \in M(X_{i_n})} A(x_n) \rightarrow \coprod_{x \in M(X_i)} A(x)$$

given by

$$\omega(a_1, \dots, a_n) = \sum_{\nu=1}^n \langle \omega, x_1, \dots, x_n, \nu \rangle a_\nu.$$

Then

$$\mathrm{Der}(M; A) \subset \prod_{\substack{i \in I \\ x \in M(X_i)}} A(x)$$

consists of those families  $(d(x) \in A(x))_{x \in \coprod_i M(X_i)}$  which respect all these operations. That is,  $\mathrm{Der}(M; A)$  consists of assignments, to each  $x \in M(X_i)$ , of an element  $d(x) \in A(x)$ , in such a way that for any  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$  and any  $x_\nu \in M(X_{i_\nu})$ ,  $\nu = 1, \dots, n$ , one has

$$(*) \quad d(\omega(x_1, \dots, x_n)) = \sum_{\nu=1}^n \langle \omega, x_1, \dots, x_n, \nu \rangle d(x_\nu).$$

Because of this expression it is natural to call such assignments *derivations*.

We then present  $\Omega_M^1$  by generators and relations as a  $\mathcal{U}(M)$ -module as follows: it has generators  $d(x) \in \Omega_M^1(x)$  for each  $x \in M(X_i)$  and each  $i \in I$ ; and the defining relations are (\*) above. It is then clear that  $\Omega_M^1$  carries a generic derivation  $d$ , so that one has a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}(M)}(\Omega_M^1, A) \xrightarrow{\cong} \mathrm{Der}(M; A)$$

given by  $f \mapsto fd$ . That  $\Omega^1$  is functorial in  $M$  is also clear from the construction.

Now suppose  $M$  is a finitely generated free model  $F(X)$ , i. e. there is an  $X \in \mathbb{A}$  with  $M = \mathrm{Hom}_{\mathbb{A}}(X, \_)$ . Then it is straightforward to check using Yoneda lemma that for an object of  $\mathbb{A}\mathbf{b}(\mathbb{A}\text{-}\mathbf{mod}/M)$  corresponding to a  $\mathcal{U}(M)$ -module  $A$  we will have  $\mathrm{Der}(F(X); A) \cong A(\mathrm{id}_X)$ . It follows that  $\mathrm{Hom}_{\mathcal{U}(F(X))}(\Omega_{F(X)}^1, A)$  is an exact functor of  $A$ , i. e.  $\Omega_{F(X)}^1$  is projective. In fact of course this actually means that  $\Omega_{F(X)}^1 = \mathrm{h}_{\mathrm{id}_X}$ .  $\square$

## 5. CARTESIAN NATURAL SYSTEMS

**5.1. Definitions, motivation, examples.** Let  $\mathbb{A}$  be a theory and let  $D$  be a natural system on  $\mathbb{A}$ . We will say that the natural system  $D$  is *cartesian* (or *compatible with products* — cf. [8]) if for any product diagram  $p_k : X_1 \times \dots \times X_n \rightarrow X_k$ ,  $k = 1, \dots, n$  and any morphism  $f : X \rightarrow X_1 \times \dots \times X_n$  the homomorphism

$$D_f \rightarrow D_{p_1 f} \times \dots \times D_{p_n f}$$

given by  $a \mapsto (p_1 a, \dots, p_n a)$  is an isomorphism. Obviously  $D$  is cartesian if and only if it satisfies the above condition with  $n = 0$  and  $n = 2$ , i. e.

- $D_{!_X} = 0$  for the unique morphism  $!_X : X \rightarrow 1$  to the terminal object;
- $D_f \rightarrow D_{p_1 f} \times D_{p_2 f}$  is an isomorphism for any  $f : X \rightarrow X_1 \times X_2$ .

One observes that if a bifunctor  $D : \mathbb{A}^{\text{op}} \times \mathbb{A} \rightarrow \mathcal{M}$  preserves products in the second variable, then the natural system induced by  $D$  is cartesian. We denote by  $\mathcal{F}(\mathbb{A})$  the category of cartesian natural systems on  $\mathbb{A}$ .

*Example 5.1.1.* Recall that in 3.3 we have defined the notion of a module over a ringoid-valued functor. Let us then, for a theory  $\mathbb{A}$ , consider the ringoid-valued functor  $\mathcal{U}_{\mathbb{A}}$  on  $\mathbb{A}$  given by  $X \mapsto \mathcal{U}_{\mathbb{A}}(F(X))$ , where  $F(X) = \text{Hom}_{\mathbb{A}}(X, \_)$  is the free finitely generated  $\mathbb{A}$ -model corresponding to the object  $X$ . For any two objects  $A, B$  of  $\mathcal{U}_{\mathbb{A}}\text{-mod}$ , similarly to the natural systems  $\mathcal{H}om$  defined in 3.1, there is a natural system  $\mathcal{H}om(A, B)$  on  $\mathbb{A}$  given by

$$\mathcal{H}om(A, B)_{X \xrightarrow{f} Y} = \text{Hom}_{\mathcal{U}_{\mathbb{A}}(F(Y))}(A_Y, F(f)^* B_X),$$

where the ringoid morphism  $F(f) : \mathcal{U}_{\mathbb{A}}(F(Y)) \rightarrow \mathcal{U}_{\mathbb{A}}(F(X))$  is induced by  $F(f) : F(Y) \rightarrow F(X)$ , i. e. by  $(g \mapsto gf) : \text{Hom}_{\mathbb{A}}(Y, \_) \rightarrow \text{Hom}_{\mathbb{A}}(X, \_)$ . Let us find out when is this natural system cartesian. For this it will be convenient to rewrite the above in the following way:

$$\mathcal{H}om(A, B)_{X \xrightarrow{f} Y} = \text{Hom}_{\mathcal{U}_{\mathbb{A}}(F(X))}(F(f)!_Y A_Y, B_X).$$

Indeed as we saw in 4.1 all the functors  $F(f)^*$  have left adjoints. The above conditions then show that this natural system is cartesian if and only if

- $\text{Hom}_{\mathcal{U}(F(X))}(F(!_X)!_1 A_1, B_X) = 0$  for all  $X$ ;
- the canonical morphism

$$\text{Hom}_{\mathcal{U}(F(X))}(F(f)!_1 A_{X_1 \times X_2}, B_X) \rightarrow \text{Hom}_{\mathcal{U}(F(X))}(F(p_1 f)!_1 A_{X_1} \oplus F(p_2 f)!_1 A_{X_2}, B_X)$$

is an isomorphism for any  $f : X \rightarrow X_1 \times X_2$ .

In particular  $\mathcal{H}om(A, B)$  is cartesian for *all*  $B$  if and only if  $A$  satisfies

- $A_1 = 0$ ;
- $F(p_1)!_1 A_{X_1} \oplus F(p_2)!_1 A_{X_2} \rightarrow A_{X_1 \times X_2}$  is an isomorphism for any  $X_1, X_2$ .

It is natural to call such an  $A$  a *cartesian  $\mathcal{U}_{\mathbb{A}}$ -module*.

We already have a nice example of such: the  $\Omega^1$  constructed above. Indeed any  $\mathcal{U}_{\mathbb{A}}$ -module  $B$  determines a natural system  $\text{Der}(\_, B)$  on  $\mathbb{A}$  in the following way: for a morphism  $f : X \rightarrow Y$  of  $\mathbb{A}$ , put

$$\text{Der}(\_, B)_f = \text{Der}(F(Y); f^*(B_X)).$$

Here  $p_X : B_X \rightarrow F(X)$  is the object of  $\mathbb{A}\text{b}(\mathbb{A}\text{-mod}/F(X))$  corresponding to  $B(X)$  under the equivalence  $\mathcal{U}_{\mathbb{A}}(F(X))\text{-mod} \simeq \mathbb{A}\text{b}(\mathbb{A}\text{-mod}/F(X))$ . That this is indeed a natural system, follows from the functorial properties of  $\text{Der}$ . Moreover this natural system is cartesian. Indeed,  $\mathbb{A}$ -models of the form  $F(X)$  are the representable ones,  $F(X)(Y) = \text{Hom}_{\mathbb{A}}(X, Y)$ . Then considering the diagram  $(\ddagger)$  we see that  $\text{Der}(F(Y); f^*(B_X))$  can be identified with the set of all elements  $b \in B_X(Y)$  with  $p_X(b) = f \in F(X)(Y) =$

$\text{Hom}_{\mathbb{A}}(X, Y)$ . Then given  $f_i : X \rightarrow X_i$ ,  $i = 1, \dots, n$ , one has

$$\begin{aligned} \text{Der}(\cdot; B)_{(f_1, \dots, f_n)} &= \text{Der}(F(X_1 \times \dots \times X_n); (f_1, \dots, f_n)^*(B_X)) \\ &\approx \{b \in B_X(X_1 \times \dots \times X_n) \mid p_X(b) = (f_1, \dots, f_n)\} \\ &\approx \{(b_1, \dots, b_n) \in B_X(X_1) \times \dots \times B_X(X_n) \mid p_X(b_i) = f_i, i = 1, \dots, n\} \\ &\approx \text{Der}(\cdot; B)_{f_1} \times \dots \times \text{Der}(\cdot; B)_{f_n}. \end{aligned}$$

But it is immediate from 4.2.6 that there is an  $\mathcal{U}_{\mathbb{A}}$ -module  $\Omega^1$  such that the natural system  $\text{Der}(\cdot; B)$  is actually isomorphic to  $\mathcal{H}om(\Omega^1_{F(\cdot)}, B)$ . Namely,  $\Omega^1$  is just given by  $X \mapsto \Omega^1_{F(X)}$ . It is then a cartesian  $\mathcal{U}_{\mathbb{A}}$ -module, i. e. one has

- $\Omega^1_{F(1)} = 0$ ;
- $F(p_1)! \Omega^1_{F(X_1)} \oplus F(p_2)! \Omega^1_{F(X_2)} \rightarrow \Omega^1_{F(X_1 \times X_2)}$  is an isomorphism for any  $X_1, X_2$ .

The following fact goes back to [13].

**Lemma 5.1.2.** *Let*

$$0 \rightarrow D \rightarrow \mathbb{E} \xrightarrow{P} \mathbb{A} \rightarrow 0$$

*be a linear extension of a theory  $\mathbb{A}$  by a natural system  $D$ . Then  $D$  is cartesian iff  $\mathbb{E}$  is a theory and  $P$  is a theory morphism.*

*Proof.* Take a product diagram  $p_i : X_1 \times \dots \times X_n \rightarrow X_i$ ,  $i = 1, \dots, n$ , and choose arbitrarily  $\tilde{p}_i$  in  $\mathbb{E}$  with  $P(\tilde{p}_i) = p_i$ . This then gives a commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbb{E}}(X, X_1 \times \dots \times X_n) & \xrightarrow{\tilde{f} \mapsto (\tilde{p}_1 \tilde{f}, \dots, \tilde{p}_n \tilde{f})} & \text{Hom}_{\mathbb{E}}(X, X_1) \times \dots \times \text{Hom}_{\mathbb{E}}(X, X_n) \\ P \downarrow & & \downarrow P \\ \text{Hom}_{\mathbb{A}}(X, X_1 \times \dots \times X_n) & \xrightarrow{\approx} & \text{Hom}_{\mathbb{A}}(X, X_1) \times \dots \times \text{Hom}_{\mathbb{A}}(X, X_n) \end{array}$$

which shows that  $\mathbb{E}$  has and  $P$  preserves finite products iff all the maps

$$P^{-1}(f) \rightarrow P^{-1}(p_1 f) \times \dots \times P^{-1}(p_n f),$$

given by  $\tilde{f} \mapsto (\tilde{p}_1 \tilde{f}, \dots, \tilde{p}_n \tilde{f})$  are bijective.

On the other hand the above maps are equivariant with respect to the group homomorphisms

$$D_f \rightarrow D_{p_1 f} \times \dots \times D_{p_n f}$$

and the actions given by the linear extension structure. Our proposition then follows from the following easy lemma.  $\square$

**Lemma 5.1.3.** *Suppose given a group homomorphism  $f : G_1 \rightarrow G_2$  and an  $f$ -equivariant map  $x : X_1 \rightarrow X_2$  between sets  $X_i$  with transitive and effective  $G_i$ -actions. Then  $x$  is bijective iff  $f$  is an isomorphism.*

*Proof.* See e. g. [13, Lemma 3.5]  $\square$

**Theorem 5.1.4.** *There is an equivalence of categories*

$$\Phi : \mathcal{F}(\mathbb{A}) \rightarrow \mathcal{U}_{\mathbb{A}}\text{-}\mathbf{mod};$$

in particular,  $\mathcal{F}(\mathbb{A})$  is an abelian category with enough projectives and injectives. Moreover the quasi-inverse of this equivalence assigns to an object  $A$  of  $\mathcal{U}_{\mathbb{A}}\text{-}\mathbf{mod}$  the cartesian natural system  $\text{Der}(\_, A)$  from 5.1.1.

*Proof.* As always, we can assume here that  $\mathbb{A}$  is an  $I$ -sorted theory. Then for a cartesian natural system  $D$  on  $\mathbb{A}$ , to define  $\Phi(D)$  we must first name for each  $X \in \text{Ob}(\mathbb{A})$  a  $\mathcal{U}_{\mathbb{A}}(F(X))$ -module  $\Phi(D)_X$ . The set of objects of  $\mathcal{U}_{\mathbb{A}}(F(X))$  is  $\coprod_{i \in I} F(X)(X_i)$  (see 4.2.3), i. e.  $\coprod_{i \in I} \text{Hom}_{\mathbb{A}}(X, X_i)$ . We then define values of  $\Phi(D)_X$  on these objects by

$$\Phi(D)_X(X \xrightarrow{x} X_i) = D_x.$$

Next action of morphisms of  $\mathcal{U}_{\mathbb{A}}(F(X))$  is uniquely determined by requiring, for  $(x_1, \dots, x_n) : X \rightarrow X_{i_1} \times \dots \times X_{i_n}$  and  $\omega : X_{i_1} \times \dots \times X_{i_n} \rightarrow X_i$ , commutativity of the diagrams

$$\begin{array}{ccc} & D_{(x_1, \dots, x_n)} & \xleftarrow{\cong} D_{x_1} \times \dots \times D_{x_n} \\ \omega_- \swarrow & & \nwarrow \iota_\nu \\ D_{\omega(x_1, \dots, x_n)} & \xleftarrow{\langle \omega, x_1, \dots, x_n, \nu \rangle} & D_{x_\nu} \end{array}$$

where the isomorphism is the inverse of the canonical map that is required by cartesianness of  $D$ , and  $\iota_\nu$  is the  $\nu$ -th embedding into  $\oplus = \times$  of abelian groups.

We also have to define action on  $\Phi(D)$  of morphisms  $f : X \rightarrow Y$  in  $\mathbb{A}$ , which must be  $\mathcal{U}_{\mathbb{A}}(F(Y))$ -module morphisms  $\Phi(D)_Y \rightarrow F(f)^*(\Phi(D)_X)$ , where the functor  $F(f)^* : \mathcal{U}_{\mathbb{A}}(F(X))\text{-}\mathbf{mod} \rightarrow \mathcal{U}_{\mathbb{A}}(F(Y))\text{-}\mathbf{mod}$  is the restriction of scalars along the ringoid morphism  $\mathcal{U}_{\mathbb{A}}(F(Y)) \rightarrow \mathcal{U}_{\mathbb{A}}(F(X))$  induced by the morphism of  $\mathbb{A}$ -models  $F(f) : F(Y) \rightarrow F(X)$ . Now  $F(f)^*(\Phi(D)_X)$  is easily seen to be given by  $(y : Y \rightarrow X_i) \mapsto D_{yf}$ , so what we must choose is a suitably compatible family of abelian group homomorphisms

$$\Phi(D)_f(Y \xrightarrow{y} X_i) : D_y \rightarrow D_{yf},$$

and these we declare to be the action of  $\_f$  on  $D$ . It is then straightforward that all of the above indeed gives a functor  $\Phi : \mathcal{F}(\mathbb{A}) \rightarrow \mathcal{U}_{\mathbb{A}}\text{-}\mathbf{mod}$ .

Next note that, as we have seen in 4.2.6, one has  $\text{Der}(F(X); A) \cong A(\text{id}_X)$  for any  $\mathcal{U}_{\mathbb{A}}(F(X))$ -module  $A$ , so in particular for any  $f : X \rightarrow Y$  in  $\mathbb{A}$  we have by 5.1.1

$$\text{Der}(\_, \Phi(D))_f = \text{Der}(F(Y); F(f)^*(\Phi(D)_X)) \cong F(f)^*(\Phi(D)_X)(\text{id}_Y) = D_{\text{id}_Y f} = D_f.$$

Conversely, given a  $\mathcal{U}_{\mathbb{A}}$ -module  $A$ , by definition

$$\begin{aligned} \Phi(\text{Der}(\_, A))_X(X \xrightarrow{x} X_i) &= \text{Der}(\_, A)_x = \text{Der}(F(X_i); F(x)^*(A_X)) \\ &\cong F(x)^*(A_X)(\text{id}_{X_i}) = A_X(x). \end{aligned}$$

(Of course one should also check these on morphisms, but this is straightforward too).  $\square$

**5.2. Cohomology of theories.** For a theory  $\mathbb{A}$  and an object  $A \in \mathcal{U}_{\mathbb{A}}\text{-mod}$ , we next define the cohomology

$$H^*(\mathbb{A}; A)$$

by the equality

$$H^*(\mathbb{A}; A) := \text{Ext}_{\mathcal{U}_{\mathbb{A}}\text{-mod}}(\Omega^1, A).$$

Here  $\Omega^1$  is turned into an object of  $\mathcal{U}_{\mathbb{A}}\text{-mod}$  as in 5.1.1 above.

The following is the main theorem of this section

**Theorem 5.2.1.** *Let  $\mathbb{A}$  be a theory and let  $A$  be an  $\mathcal{U}_{\mathbb{A}}$ -module. Then*

$$H^*(\mathbb{A}; A) \cong H^*(\mathbb{A}; \text{Der}(\_, A)),$$

where on the left we have cohomology just defined, while on the right — the Baues-Wirsching cohomology of the category  $\mathbb{A}$  with coefficients in the natural system  $\text{Der}(\_, A)$ .

*Proof.* By Proposition 4.2.6 one has an isomorphism of natural systems:

$$\text{Der}(\_, A) \cong \mathcal{H}em(\Omega^1, A).$$

Hence the result is a consequence of Corollary 3.1.3. The fact that the condition of Corollary 3.1.3 holds follows from Proposition 4.2.6.  $\square$

**Corollary 5.2.2.** *If  $\mathbb{F}$  is a free  $I$ -sorted theory and  $D$  is a cartesian natural system on  $\mathbb{F}$ , then*

$$H^i(\mathbb{F}; D) = 0, \quad i > 1.$$

*Proof.* First consider the case  $i = 2$ ; thanks to Theorem 2.5.1 it suffices to show that any linear extension of  $\mathbb{F}$  by  $D$  splits. By Lemma 5.1.2 any such extension is an extension in  $\mathcal{Theories}$  and we can use Proposition 4.1.3 to conclude that it really splits. If  $i \geq 3$  we can use Theorem 5.2.1 to pass to the theory cohomologies. The latter are Ext-groups in appropriate abelian categories vanishing on injective objects and in dimension two we can use the long cohomological sequence associated to an extension

$$0 \rightarrow A \rightarrow I \rightarrow B \rightarrow 0$$

with injective  $I$  to finish the proof.  $\square$

This result in the case when  $D$  is a bifunctor over a single sorted theory was proved in [13] (see Proposition 4.22 of loc. cit.).

## 6. TRACK THEORIES AND THE PROOF OF THE MAIN THEOREM

**6.1. Track theories.** Recall that a track category  $\mathcal{T}$  is said to have *finite lax products*, if for any finite collection  $X_1, \dots, X_n$  of its objects there exists a family of 1-arrows  $p_1 : X \rightarrow X_1, \dots, p_n : X \rightarrow X_n$  such that the induced functors

$$\llbracket Y, X \rrbracket \rightarrow \llbracket Y, X_1 \rrbracket \times \dots \times \llbracket Y, X_n \rrbracket,$$

$f \mapsto (p_1 f, \dots, p_n f)$ , are equivalences of groupoids for any object  $Y$  of  $\mathcal{T}$ . A track category with finite lax products is called a *track theory*. A track theory is called *strong* if the

products in it are in fact strong, i. e. the above equivalences of groupoids are in fact isomorphisms.

As we saw a linear extension of a theory  $\mathbb{A}$  by a natural system  $D$  is again a theory provided  $D$  is cartesian. The situation changes dramatically for track extensions. Let  $\mathcal{T}$  be a linear track extension of a theory  $\mathbb{A}$  by a cartesian natural system  $D$ . Then in general  $\mathcal{T}$  is not a strong track theory, but only a track theory. This is the subject of the following

**Proposition 6.1.1.** *For a linear track extension*

$$D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \rightarrow \mathbb{A},$$

*the corresponding track category is a track theory if and only if  $\mathbb{A}$  is a theory and  $D$  is a cartesian natural system.*

*Proof.* Assume  $\mathcal{T}$  is a track theory, so that

$$[[Y, X]] \rightarrow [[Y, X_1]] \times \dots \times [[Y, X_n]]$$

are equivalences of groupoids. Thus they induce bijections on connected components and therefore  $\mathcal{T}_\simeq \cong \mathbb{A}$  is a theory. The same equivalence induces an isomorphism of groups

$$\text{Aut}(f) \rightarrow \text{Aut}(f_1) \times \dots \times \text{Aut}(f_n),$$

where  $f : Y \rightarrow X$  is an 1-arrow in  $\mathcal{T}$  and  $f_i = p_i f : Y \rightarrow X_n$ . This fact together with the definition of a track extension shows that  $D$  is cartesian. Conversely, the above argument actually shows that the functors

$$[[Y, X]] \rightarrow [[Y, X_1]] \times \dots \times [[Y, X_n]]$$

induce bijections on connected components and induce isomorphisms of corresponding automorphism groups. Hence these functors are equivalences.  $\square$

Let  $\mathbb{A}$  be a theory and let  $D$  be a natural system. As we saw any object of the category  $\text{Trex}(\mathbb{A}; D)$  is a track theory. We now show that any morphism of  $\text{Trex}(\mathbb{A}; D)$  carries lax products to lax products. Indeed, let  $F : \mathcal{T} \rightarrow \mathcal{T}'$  be a morphism of the category  $\text{Trex}(\mathbb{A}; D)$  and suppose all

$$[[Y, X]]_{\mathcal{T}} \rightarrow [[Y, X_1]]_{\mathcal{T}} \times \dots \times [[Y, X_n]]_{\mathcal{T}}$$

are equivalences of groupoids. We have to show that then the functors

$$[[FY, FX]]_{\mathcal{T}'} \rightarrow [[FY, FX_1]]_{\mathcal{T}'} \times \dots \times [[FY, FX_n]]_{\mathcal{T}'},$$

induced by  $F$  are equivalences as well. But this is a consequence of the following fact: For any  $A, B \in \mathcal{T}$  the functor  $F_{A,B} : [[A, B]]_{\mathcal{T}} \rightarrow [[FA, FB]]_{\mathcal{T}'}$  is an equivalence of groupoids. To see the last assertion, it suffices to note that the sets of components of both groupoids in question are canonically isomorphic to  $\text{Hom}_{\mathbb{A}}(A, B)$  and  $F$  is compatible with it and also for any map  $f : A \rightarrow B$  the groups  $\text{Aut}_{\mathcal{T}}(f)$  and  $\text{Aut}_{\mathcal{T}'}(Ff)$  both are canonically isomorphic to  $D_{qf}$ . Here  $q : \mathcal{T}_0 \rightarrow \mathbb{A}$  is the canonical functor. According to Lemma 2.3.5 any morphism in  $\text{Trex}(\mathbb{A}; D)$  is an equivalence of theories.



**6.2. The main theorem.** We let  $\text{Str}(\mathbb{A}; D)$  be the full subcategory of  $\text{Trext}(\mathbb{A}; D)$  whose objects are strong track theories. There is an obvious functor

$$\text{Str}(\mathbb{A}; D) \rightarrow \text{Trext}(\mathbb{A}; D).$$

Our Theorem 6.2.1 shows that it yields a bijection on the set of connected components.

The proof of Theorem 6.2.1 uses the following useful construction. Let

$$0 \rightarrow D \rightarrow \mathcal{T}_1 \rightrightarrows \mathcal{T}_0 \xrightarrow{p} \mathbf{C} \rightarrow 0$$

be a linear track extension of a small category  $\mathbf{C}$  by a natural system  $D$ . Here  $D$  is any natural system on  $\mathbf{C}$ . Suppose a functor  $f : \mathbf{E} \rightarrow \mathcal{T}_0$  is given which is identity on objects. We assume that the composition  $pf : \mathbf{E} \rightarrow \mathbf{C}$  is full. We construct now a linear track extension

$$0 \rightarrow D \rightarrow \mathcal{T}'_1 \rightrightarrows \mathcal{T}'_0 \xrightarrow{pf} \mathbf{C} \rightarrow 0$$

with the property  $\mathcal{T}'_0 = \mathbf{E}$  and a morphism of linear track extensions  $\mathcal{T} \rightarrow \mathcal{T}'$  in  $\text{Trext}(\mathbf{C}; D)$ . If  $x, y : A \rightarrow B$  are in  $\mathbf{E}$ , then there exists a track  $x \Rightarrow y$  in  $\mathcal{T}'$  iff  $pf(x) = pf(y)$ . If this holds, then we define the set  $\text{Hom}_{[[A, B]]_{\mathcal{T}'}}(x, y)$  to be  $\text{Hom}_{[[A, B]]_{\mathcal{T}}}(fx, fy)$ . This defines a track extension  $\mathcal{T}'$  which is denoted also by  $f^!(\mathcal{T})$ . The track functor  $f^!(\mathcal{T}) \rightarrow \mathcal{T}$  is identity on objects, on maps it is given by  $f$  and on tracks it is the inclusion.

**Theorem 6.2.1.** *Let  $\mathbb{A}$  be a theory and let  $D$  be a cartesian natural system on  $\mathbb{A}$ . Then there exists a bijection*

$$\pi_0(\text{Str}(\mathbb{A}; D)) \cong H^3(\mathbb{A}; D).$$

*Proof.* Let  $\mathcal{T}$  be an object of the category  $\text{Str}(\mathbb{A}; D)$ . Then it can be also considered as an object of  $\text{Trext}(\mathbb{A}, \mathcal{T}_0; D)$  and therefore it defines an element in  $H^3(\mathbb{A}, \mathcal{T}_0; D)$  thanks to Theorem 2.3.7. Then applying to this element the boundary homomorphism  $H^3(\mathbb{A}, \mathcal{T}_0; D) \rightarrow H^3(\mathbb{A}; D)$  gives an element in  $H^3(\mathbb{A}; D)$ . In this way we get a map

$$\xi : \pi_0(\text{Str}(\mathbb{A}; D)) \rightarrow H^3(\mathbb{A}; D).$$

We have to show that this map is a bijection. Take an  $a \in H^3(\mathbb{A}; D)$ . There is a free theory  $\mathbb{F}$  and a morphism of theories  $r : \mathbb{F} \rightarrow \mathbb{A}$  which is a full functor. Thanks to Theorem 5.2.1 we have  $H^i(\mathbb{F}; D) = 0$  for all  $i \geq 2$ . Therefore the connecting homomorphism

$$\partial : H^3(\mathbb{A}, \mathbb{F}; D) \rightarrow H^3(\mathbb{A}; D)$$

is an isomorphism. Let  $b = \partial^{-1}(a) \in H^3(\mathbb{A}, \mathbb{F}; D)$  be the element corresponding to  $a$ . Thanks to Theorem 2.3.7 the element  $b$  defines a track extension  $\mathcal{T}$  with  $\mathcal{T}_0 = \mathbb{F}$ . Thus  $\mathcal{T}$  is a strong track theory and hence  $\xi$  is surjective. It remains to show that  $\xi$  is injective. Suppose  $\xi(\mathcal{T}) = \xi(\mathcal{T}')$ . Since  $p : \mathcal{T} \rightarrow \mathbb{A}$  and  $p' : \mathcal{T}' \rightarrow \mathbb{A}$  are full morphisms of theories, one can lift the morphism of theories  $r : \mathbb{F} \rightarrow \mathbb{A}$  to morphisms  $q : \mathbb{F} \rightarrow \mathcal{T}_0$  and  $q' : \mathbb{F} \rightarrow \mathcal{T}'_0$  of theories, where  $\mathbb{F}$  is a free theory and  $r$  is surjective. Using the  $(\_)^!$ -construction one obtains the linear track extensions  $q^!(\mathcal{T})$  and  $q'^!(\mathcal{T}')$  together with morphisms of linear track extensions  $q^!(\mathcal{T}) \rightarrow \mathcal{T}$  and  $q'^!(\mathcal{T}') \rightarrow \mathcal{T}'$ . Now both  $q^!(\mathcal{T})$  and  $q'^!(\mathcal{T}')$  lie in  $\text{Trext}(\mathbb{A}, \mathbb{F}; D)$ , and their classes in  $H^3(\mathbb{A}, \mathbb{F}; D)$  coincide with images of the classes of  $\mathcal{T}$  and  $\mathcal{T}'$  under the homomorphisms  $q^* : H^3(\mathbb{A}, \mathcal{T}_0; D) \rightarrow H^3(\mathbb{A}, \mathbb{F}; D)$

and  $q^* : H^3(\mathbb{A}, \mathcal{T}'_0; D) \rightarrow H^3(\mathbb{A}, \mathbb{F}; D)$  respectively. It follows from our assumptions that these classes are the same and therefore  $q^!(\mathcal{T})$  and  $q^!(\mathcal{T}')$  are isomorphic in the groupoid  $\text{Trext}(\mathbb{A}, \mathbb{F}; D)$ . Therefore we have the following diagram in  $\text{Str}(\mathbb{A}; D)$ :

$$\mathcal{T}' \leftarrow q^! \mathcal{T}' \cong q^! \mathcal{T} \rightarrow \mathcal{T}$$

and hence the result.  $\square$

We are now in a position to prove our main result:

**Theorem 6.2.2.** *Any abelian track theory is equivalent to a strong one. More precisely if  $\mathcal{T}$  is an abelian track theory, then there exists a strong abelian track theory  $\mathcal{T}'$ , an abelian track theory  $\mathcal{T}''$ , and weak equivalences  $\mathcal{T} \leftarrow \mathcal{T}'' \rightarrow \mathcal{T}'$  as well as a lax equivalence  $\mathcal{T}' \rightarrow \mathcal{T}$ .*

*Proof.* Let  $\mathcal{T}$  be an abelian track theory. Then the corresponding homotopy category  $\mathbb{A} := \mathcal{T}_\sim$  is a theory. Since any abelian track category is part of a linear track extension, there is a natural system  $D$  on  $\mathbb{A}$  such that  $\mathcal{T} \in \text{Trext}(\mathbb{A}; D)$ . By Proposition 6.1.1  $D$  is a cartesian natural system and therefore we can use Theorem 6.2.1 to show that there is an expected path in  $\text{Trext}(\mathbb{A}; D)$  connecting  $\mathcal{T}$  to an object of  $\text{Str}(\mathbb{A}; D)$ . All maps in these diagrams are weak equivalences thanks to Lemma 2.3.5. The last assertion follows from Theorem 2.4.2.  $\square$

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